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INVARIANT MEASURES ON LOCALLY COMPACT SPACES

BY

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1. Introduction. This paper is concerned with Haar measure on locally compact groups, and more generally, with group-invariant measures on locally compact spaces. An attempt has been made to present a complete and fairly self-contained account of the fundamentals of this subject, though it has been necessary to assume a knowledge, on the part of the reader, of the basic results of measure theory. It is shown in the following sections that an equicontinuous group of unimorphisms of a uniformly locally compact space leaves invariant a non-zero regular measure on the space, and that such a measure is unique if and only if there is a point in the space whose orbit is dense. A special case is the existence and uniqueness of Haar measure, and our proof of the more general result resembles known proofs, notably that of Haar (as modified by Weil and Kodaira for inseparable groups, and utilizing measure-theoretic notions originating with von Neumann—unpublished in part—with regard to the existence, and that of Weil, as regards the uniqueness, of the invariant measure). A by-product is a proof that a positive linear functional, on the class of all those continuous functions on a locally compact Hausdorff space which vanish outside compact sets, is given by an integral relative to a regular measure.

Many of the further properties of invariant measures involve consideration of integrals ("convolutions") of the form $\int_G f(a^{-1}p)\, g(a)\, da$ over a locally compact group G, and we give a simple direct proof that such integrals exist, for appropriate f and g; this work likewise applies

to the case of a group acting on space, when f is taken to be a function on the space. An important tool here is (a slight modification of) Stone's generalization of the Weierstrass approximation theorem. This kind of generalized approximation theorem has previously been used by Bourbaki in a development (unpublished) of the theory of products of measures on locally compact spaces.

2. Definitions and notations. The present section gives our definitions of various terms, mainly relating to measure theory, which we use in later sections. The concept of σ-finite measure space, and especially the demonstration of its utility in formulating the Fubini theorem, are due to P. R. Halmos. Although we make no use of Banach-space-valued functions in treating invariant measures, we give some consideration to such functions in this section and the following two, partly because of their usefulness in some aspects of the theory of invariant measures, and partly because of the lack of a full treatment in the literature of the features considered here.

Definitions 1. A *σ-field* is a collection of subsets of a set, which is closed under countable unions, intersections, and relative complements. A *measure* is a countably-additive non-negative-valued function ($+\infty$ being an allowed value) on a σ-field. A measure space (S, F, μ) is the combination of a set S, a σ-field F of subsets of S, and a measure μ on F. This measure space is complete if every subset of a set in F of measure zero is likewise in F; it is called *σ-finite* if every element of F is σ-finite (i.e. is a countable union of elements of F of finite measure). The space of αth power complex-valued integrable functions over the space is denoted by $L_\alpha(S, F, \mu)$, or for brevity by $L_\alpha(S, \mu)$.

Definitions 2. A regular measure μ on a locally compact Hausdorff space S is a complete, σ-finite, measure space (S, F, μ) such that: (*a*) F contains all

compact subsets of S, (b) for any $E \varepsilon F$, there exists an open set $O \varepsilon F$ containing E, and $\mu(E) = LUB\ [C \subset E, C \text{ compact}]\ \mu(C) = GLB\ [E \subset O, O \text{ open}, O \varepsilon F]\ \mu(O)$.

We mention that the requirements that the measure space be complete and σ-finite are mainly for convenience; if non-complete the space could be completed in a simple fashion by enlarging F, and if non-σ-finite, it could be made such by removing the non-σ-finite sets from F.

DEFINITIONS 3. A set Z in a complete measure space (S, F, μ) (i.e. a subset of S) is *equivalent to zero* if for every $E \varepsilon F$, $E \cap Z$ is a set of measure zero. Two functions f and g on S are *equivalent* or equal *nearly everywhere* (n.e.) if $f(s) = g(s)$ except for a set of s-values equivalent to zero. Two subsets of S, P and Q, are *equivalent* if their symmetric difference $(P-Q) \cup (Q-P)$ is equivalent to zero. A sequence $\{ f_n \}$ of functions on S *converges nearly everywhere* to a function f on S if $\lim_{n \to \infty} f_n(s)$ exists except for a set Z_1 of s-values equivalent to zero, and if there is a set Z_2 equivalent to zero, such that $\lim_{n \to \infty} f_n(s) = f(s)$, for S not in $Z_1 \cup Z_2$; f is called a *limit nearly everywhere* of the sequence. A function f on S to a Banach space is measurable if it is a limit nearly everywhere of linear combinations of characteristic functions of sets in F.

DEFINITIONS 4. If (S_j, F_j, μ_j) $(j = 1, 2)$ are two σ-finite measure spaces, the *product measure space* is the unique σ-finite complete measure space $S, F, \mu)$ with the following properties: $S = S_1 \times S_2$, F contains all product sets $E_1 \times E_2$ with $E_j \varepsilon F_j$, μ satisfies $\mu(E_1 \times E_2) = \mu_1(E_1)\ \mu_2(E_2)$ if E_j is an element of F_j of finite measure, and every set in F differs by a set of measure zero from a set in the σ-field generated by the products $E_1 \times E_2$, with $E_j \varepsilon F_j$. (See(3) or(7).)*

* Numbers in brackets denote references at the end of this paper.

DEFINITIONS 5. A *topological group G acts continuously on a topological space M* if for each $a \varepsilon G$ there is a homeomorphism ϕ_a of M, such that $\phi_a(p)$ is a continuous function of (p, a) on the product topological space $M \times G$, and if $\phi_{ab} = \phi_a \phi_b$ for all a and b in G. (For brevity, $\phi_a(p)$ is also written simply as ap). A *unimorphism* of a uniform space (see[10]) is a homeomorphism which, together with its inverse, is uniformly continuous. A group G of unimorphisms of a uniform space is *equicontinuous* if, putting $\mathcal{N}_\sigma(p)$ for the family of neighbourhoods with index σ, for any σ there is a ρ such that $\phi_a(\mathcal{N}_\rho(p)) \subset \mathcal{N}_\sigma(\phi_a(p))$ for all $a \varepsilon G$ and $p \varepsilon M$. A transformation $s \rightarrow \psi(s)$ on a complete measure space (S, F, μ) is *measure-preserving* if $E \varepsilon F$ implies that $\psi(E)$ is in F and $\mu(E) = \mu(\psi(E))$, and if moreover the same is true of ψ^{-1}.

3. CRITERIA FOR MEASURABILITY. We shall take the basic results of measure theory for granted in the following* but include proofs of the next two theorems, which we need in later sections. For the case of complex-valued functions on a σ-finite measure space (S, F, μ) in which $S \varepsilon F$, these theorems are standard, and for Banach-space-valued functions on the same kind of measure space they are due to Dunford.

THEOREM 1. *A function f on a complete measure space (S, F, μ) to a Banach space B is measurable if and only if: (a) for each closed set K excluding the origin, $f^{-1}(K)$ is equivalent to a set in F, (b) f is nearly separably-valued (i.e. there is a set Z equivalent to zero such that $f(S-Z)$ is separable).*

Suppose first that f is measurable, so that for s not in Z, Z being equivalent to zero, $f(s) = \lim_{n\to\infty} f_n(s)$, where f_n is a linear combination of characteristic functions of sets in F. Since each f_n is finitely-valued, and since $f(S-Z)$ is in the closure of the union of the ranges of the f_n $(n = 1,2,\ldots)$, f must be almost separably-valued.

* See *e.g.*, Ch. I and II of (7), or the forthcoming book on measure theory by P. R. Halmos.

Now if K is any closed set, let K_m be the set of points whose distance from K is not greater than m^{-1}. It is straightforward to verify that

$$[s \mid f(s) \varepsilon K, \ s \text{ not in } Z] = \bigcap_{m \geqslant m_0} \bigcup_{p \geqslant 1} \bigcap_{n \geqslant p} [s \mid f_n(s) \varepsilon K_m, \ s \text{ not in } Z]$$

for any m_0. Choosing m_0 so that m_0^{-1} is less than the distance from K to the origin, the sets $[s \mid f_n(s) \varepsilon K_m, \ s$ not in $Z]$ are in F and hence so is $f^{-1}(K)$.

Next, suppose that f satisfies (a) and (b). Put $\{x_p ; p = 1, 2, \ldots\}$ for a dense subset of $f\ (S-Z)$ and $x_0 = 0$. Let $K_{n,p}$ be a sphere of radius n^{-1} and center x_p. We define g_n inductively by: $g_n(s) = 0$ for $s \varepsilon f^{-1}(K_{n,0})$; $g_n(s) = x_1$ for $s \varepsilon f^{-1}\ (K_{n,1} - K_{n,0})$; etc. Then it is plain that g_n is well-defined on $S-Z$, that g_n is countably-valued and assumes each value other than zero on a set in F (note that the complement of $K_{n,0}$ can easily be expressed as a countable union of closed sets disjoint from the origin), and that $\| g_n(s) - f(s) \| < n^{-1}$, for $s \varepsilon S-Z$. Now let E_m be a monotone increasing sequence of sets in F of finite measure such that f and all the g_n vanish outside their union. Set $h_n = g_n\ \chi_n$, where χ_n is the characteristic function of E_n. Then it is clear that the sequence $\{h_n\}$ converges to f n.e., and that h_n vanishes outside a set of finite measure. Let $\{a_{n,m}\}$ be the nonzero values of h_n, and let $S_{n,m}$ be the set on which S_n has one of the values $a_{n,1}, \ldots, a_{n,m}$. Then for fixed n, $S_{n,m}$ increases monotonely as $m \to \infty$ to the set on which h_n is not zero, hence there is an integer p such that $S_{n,p}$ differs by a set of measure less than 2^{-n} from that set. Now putting k_n for the product of h_n with the characteristic function of $S_{n,p}$, it is clear that k_n is finitely-valued (and so a linear combination of characteristic functions of sets) and assumes each of its nonzero values on sets in F, and to conclude the proof it is only necessary to show that k_n converges to f n.e. If x is not in the union, for $n \geqslant n_0$ of the set

discarded above at the nth stage, i.e., the difference between the set on which h_n is nonzero and $S_{n,p}$, then it is plain that $k_n(x) \to f(x)$. Hence $k_n(x) \to f(x)$ for all x in $S-Z$ except those in the intersection over $n_0 \geqslant 1$ of the above union, but this set is of measure zero.

THEOREM 2. *A limit nearly everywhere of measurable functions is measurable.*

If $f(s) = \lim_{n\to\infty} f_n(s)$ except on a set Z equivalent to zero, where f_n is a linear combination of characteristic functions of sets in F, then it is plain that $f(S-Z)$ is separable, so that f satisfies Theorem 1 (*b*). Now if K is a closed set in B excluding the origin, then as before,

$$f^{-1}(K) \cap (S-Z) = \bigcap_{m\geqslant m_0} \bigcup_{p\geqslant 1} \bigcap_{n\geqslant p} [s \mid f_n(s) \in K_m, s \text{ not in } Z],$$

where K_m and m_0 are as in the proof of Theorem 1, and it follows that condition (*a*) of that theorem is likewise satisfied.

4. BAIRE FUNCTIONS ON LOCALLY COMPACT SPACES. The following definition of a Baire function (of finite class) differs from the more usual one in that we do not begin with all continuous functions, but with the subclass of those continuous functions which vanish outside of compact sets; the use of this latter class is essential for the success of our method.

DEFINITION 6. The nth *Baire class* (n a positive integer) of (Banach-space valued) functions on a locally compact Hausdorff space M is defined inductively as follows: the first Baire class consists of the pointwise limits (everywhere) of functions on M which are continuous and vanish outside compact sets; the nth Baire class ($n > 1$) consists of the functions on M which are pointwise limits of functions in the $(n-1)$th Baire class.

THEOREM 3. *A Banach-space-valued function on a regular locally compact measure space is measurable if and only if it is equivalent to a Baire function of a finite class.*

Let (M, F, μ) be a regular locally compact measure space. If f is a continuous function on M to a Banach space B, if f vanishes outside a compact set C, and if K is a closed set in B excluding the origin, then $f^{-1}(K)$ is a closed subset of C, and hence is compact. Thus (*a*) of Theorem 1 is satisfied. Now $f(M) \subset f(C) \cup \{0\}$, and $f(C)$ is compact and metric, and hence separable. Thus (*b*) of Theorem 1 is valid, and it follows that f is measurable. That every function equivalent to a Baire function is measurable can now be concluded from Theorem 2.

To show that if f is measurable, then it is equivalent to a Baire function, we observe that, as shown in the proof of Theorem 1, there is a set Z equivalent to zero such that f can be approximated uniformly outside Z, by a countably-valued function, each of whose values, with the possible exception of the value zero, is assumed on a set equivalent to one in F. It follows that it is sufficient to consider the case in which f is the characteristic function of a set E in F. Now E is σ-finite, and hence contained in a countable union of open sets, each of which is in F and has finite measure. Utilizing the regularity of the measure, it results that it is sufficient to consider the case when E is open, as well as of finite measure. Let C_n be a compact subset of E such that $\mu(E-C_n) < n^{-1}$, and let f_n be a continuous function on M which is identically one on C_n, vanishes outside an open set containing C_n and contained in E and of compact closure (that such an open set exists follows readily from the local compactness of M), and such that all of its values are in the closed interval from zero to one. Putting g_m for the product of the f_n with $n < m$, it is clear that $\{g_m\}$ is a monotone decreasing sequence of continuous functions vanishing outside compact sets, which converges n.e. to the characteristic function of E. The pointwise limit of $\{g_m\}$ is then a Baire function equal n.e. to this characteristic function.

5. Convolutions. In this section it is shown that when a locally compact group acts continuously on a locally compact space, and if there are regular invariant measures on both the group and the space (the group measure being invariant under left translations, we show in later sections that such a "Haar" measure always exists), then the "convolution" of an appropriate function on the space with an integrable function on the group exists, and tends to belong to the same class of functions as the function on the space. The convolution could also be defined as a vector-valued integral, and in this definition the preceding section is not needed, but for some purposes the representation of the convolution as an ordinary (complex-valued) integral is useful.

Definition 7. A *Haar measure* on a locally compact group G is a regular measure which is invariant when the group is regarded as acting on itself by left translation, i.e. by the mappings ϕ_a, $a \in G$, where ϕ_a is given by the equation $\phi_a(b) = ab$, $b \in G$, and for which the σ-field of definition is the completion of that generated by the compact subsets of G.

Theorem 4. *Let the locally compact group G act continuously on the locally compact space M. Let (M, F, μ) be a regular measure space invariant under G, and let γ be a Haar measure on G. Then if $g \in L_1(G, \gamma)$ and if $f \in L_\alpha(M, \mu)$ $(1 \leqslant \alpha \leqslant \infty)$, the integral*

$$\int_G f(a^{-1}p)\, g(a)\, d\gamma(a) = h(p)$$

exists for almost all $p \in M$, and the function h thereby defined is in $L_\alpha(M, \mu)$, with norm not greater than $\|g\|_1 \|f\|_\alpha$.

We consider only the case $\alpha = 1$, for the more general case follows in the same way, except that Holder's inequality must be used. The initial step is to show that every Baire function on the topological product space $M \times G$ is measurable relative to the product measure space. To show this, let u and v be complex-valued continuous functions vanishing outside compact sets, on M and G

respectively. Then the function w on $M \times G$ defined by the equation $w(p, a) = u(p)v(a)$ is measurable by Theorem 1. Since sums, products, and scalar multiples of measurable functions are likewise measurable, the algebra generated by functions on $M \times G$ of the same form as w consists of measurable functions. By Theorem 2, the closure of this algebra, relative to uniform convergence, likewise consists of measurable functions. According to the Weierstrass-Stone theorem as formulated in[1], the closure relative to uniform convergence, of the algebra generated by a collection of complex-valued continuous functions vanishing at infinity on a locally compact Hausdorff space (a function vanishes at infinity on such a space if for every positive namber ε the set of points where the function exceeds ε in absolute value is compact), consists of all such functions on the space, provided: (a) the collection is closed under complex conjugation: (b) for any ordered pair of points of the space there is a function in the collection which vanishes at the first point but not at the second. These provisions are plainly satisfied by the class of all such functions as w, and hence every complex-valued continuous function on $M \times G$ which vanishes at infinity is measurable. It follows from Theorem 2 that every Baire function on $M \times G$ is measurable.

By Theorem 3, f and g are equivalent to Baire functions f_0 and g_0 respectively. Defining $k(p, a) = f_0(p)\, g_0(a)$, it is easy to see that k is a Baire function on $M \times G$. Since the mapping $(p, a) \rightarrow (a^{-1}p, a)$ is a homeomorphism of $M \times G$ with itself, the mapping takes k into a Baire function. In other words, the function m on $M \times G$ defined by the equation $m(p, a) = f_0(a^{-1}p)\; g_0(a)$ is a Baire function. As just shown, m is measurable. Moreover, it is easily seen that the integral $\int_G \left\{ \int_M |m(p, a)|\, d\mu(p) \right\} d\gamma(a)$ is finite. Hence, by Fubini's theorem the integral $\int_G m(p, a)\, d\gamma(a) = h(p)$ eixists for almost all p, and defines

an integrable function h. Thus $\int_G f_0(a^{-1}p)\, g_0(a)\, d\gamma(a)$ exists for nearly all p. But from the equivalence of f with f_0 and of g with g_0 it results that $\int_G f_0(a^{-1}p)\, g_0(a)\, d\gamma(a) = \int_G f(a^{-1}p)\, g(a)\, d\gamma(a)$, so that the latter integral exists for nearly all p, and defines an integrable function h. To obtain the bound for $\|h\|$, we observe that $\int_M |h(p)|\, d\mu(p) \leqslant \int_M \{ \int_G |f_0(a^{-1}p)\, g_0(p)|\, d\gamma(p) \}\, d\mu(p)$, and that by an application of Fubini's theorem the last integral is simply $\|f\|_1 \|h\|_1$.

6. Existence of invariant measures. Next we show that an equicontinuous group of unimorphisms of a uniformly locally compact space (*i.e.*, a uniform space in which the uniform neighbourhood families $\mathcal{N}_\sigma$ are or can be taken to be, such that $\mathcal{N}_0(p)$ has a compact closure for every p) admits a non-vanishing invariant regular measure on the space. The special case in which the group and the space are the same, the group acting on itself by left translations, plainly establishes the existence of a Haar measure of the group. The lengthiest part of the proof is measure-theoretic in character, and establishes at the same time most of the representation theorems (Theorem 6) for positive linear functionals over the collection of continuous real-valued functions vanishing outside compact sets in a locally compact Hausdorff space. We follow the original approach of Haar, in which an invariant set function is sought, as this approach seems more intuitional and direct than the alternative one which involves finding an invariant linear functional; the two approaches are technically very similar, and by the representation theorem just quoted, yield equivalent results.

Theorem 5. *Let G be an equicontinuous group of unimorphisms of a uniformly locally compact structure M. Then there exists a regular measure on M which is invariant under G and which is not identically zero.*

The non-trivial part of the proof is divided into two lemmas, the second of which concerns the derivation of a measure from a functional on the compact subsets of a locally compact space (the group of unimorphisms plays no role here), and provides the major part of the proof of Theorem 6.

LEMMA 5.1. *There exists a real, non-negative-valued function* λ *on the collection* Ω *of all compact subsets of* M, *which is not identically zero, and has the following properties (here* C *and* D *are arbitrary elements of* Ω, *and* a *is an arbitrary element of* G): (1) $\lambda(C \cup D) \leqslant \lambda(C)+\lambda(D)$; (2) *if* $C \cap D = 0$, *then* $\lambda(C \cup D) = \lambda(C)+\lambda(D)$; (3) $\lambda(aC) = \lambda(C)$.

We begin the proof of the lemma by showing that it is sufficient to prove it for the case in which G is *weakly transitive* on M, which we define to be the situation when the translates of any non-void open set cover M (it can be shown that an equivalent condition is that there exist a point in M whose orbit is dense). Let p be any point of M and let P be the closure of Gp. Then P is clearly uniformly locally compact, in the relative uniform structure, in which a family of neighborhoods $N'_\sigma(x)$ relative to P is given by the equation $N'_\sigma(x) = N_\sigma(x) \cap P$. It is easily seen that P is invariant under G so that G induces a group of unimorphisms of P. Moreover, the induced group is likewise equicontinuous, for from the relation $aN_\rho(x) \subset N_\sigma(ax)$ $(a \in G, x \in M)$ it follows, by intersecting with P, that $aN_\rho'(x) \subset N_\sigma'(ax)$ $(a \in G, x \in P)$. Now if it can be shown that G is weakly transitive on P, and if the lemma is known to be true for weakly transitive groups, then the lemma can be established in general by setting λ_0 for the function on the compact subsets of P, which is given by the lemma, and defining on the compact subsets of M by the equation $\lambda(C) = \lambda(C \cap P)$, $C \in \Omega$. That λ satisfies the stated conditions follows at once from the fact that λ_0 does.

To conclude the reduction to the case of a weakly transitive group, it remains only to show that G is weakly

transitive on P. That is, it must be shown that if O is any open set in M such that $O \neq P$ ($O \neq P$ means that O intersects P), then $G(O \cap P) \supset P$. It is straightforward to verify that $G(O \cap P) = GO \cap P$, and hence it is enough to show that $GO \supset P$. Observe first that $O \neq Gp$, whenever $O \neq P$. For let q be any point in $O \cap P$, and let $N(q)$ be any neighborhood of q such that $N(q) \subset O$. Since every neighborhood of q meets Pp, $N(q) \neq Gp$, whence $O \neq Gp$. Now let r be any point in P; we complete the argument by assuming $O \neq P$, and showing that then $r \in GO$. For every σ, $N_\sigma(r) \neq Gp$, say $b_\sigma p \in N_\sigma(r)$. It follows that $r \in (N_\sigma)^{-1}(b_\sigma p)$ (where $(N_\sigma)^{-1}$ is defined by the equation $(N_\sigma)^{-1}(x) = [y \mid x \in N_\sigma(y)]$). Now let s be any point of $O \cap Gp$, say $s = cp$. Choose σ_1 so that $N_{\sigma_1}(s) \subset O$, then choose σ_2 so that $aN_{\sigma_2}(x) \subset N_{\sigma_1}(ax)$ for $a \in G$ and $x \in M$, and finally choose σ_3 so that $(N_{\sigma_3})^{-1}(x) \subset N_{\sigma_2}(x)$ for $x \in M$. We have $r \in (N_{\sigma_3})^{-1}(b_{\sigma_3} p) \subset N_{\sigma_2}(b_{\sigma_3} p) = N_{\sigma_2}(b_{\sigma_3} c^{-1}s) \subset b_{\sigma_3} c^{-1} N_{\sigma_1}(s)$ (note that $N_{\sigma_2}(ax) \subset aN_{\sigma_1}(x)$ follows by replacing x by $a^{-1}x$), which is contained in $b_{\sigma_3} c^{-1} O \subset GO$.

From this point on in the proof of the lemma it is assumed that G is weakly transitive on M. It follows that if N is any non-void open set and C any compact set in M, then there is a least natural number, which we designate as $n(C, N)$, such that C can be covered by that many G-translates of N. In other words, $C \subset \bigcup_{j=1,\ldots,n} a_j N$, with the a_j in G and $n = n(C, N)$, and C cannot be so included for any smaller value of n. Now let K be an arbitrary but fixed non-void open set with compact closure, and put $\omega(C, N) = n(C, N)/n(K, N)$.

Next we show that ω has the following properties (here C and D are arbitrary compact sets, and N is an arbitrary non-void open set): (1) $\omega(C \cup D, N) \leq \omega(C, N) + \omega(D, N)$; (2) if $C \| D$ ($C \| D$ means that $C \cap D$ is empty) there exists a non-void open set N_0 such that

$\omega(C \cup D, \mathcal{N}) = \omega(C, \mathcal{N}) + \omega(D, \mathcal{N})$ for $\mathcal{N} \subset \mathcal{N}_0$; (3) $\omega(aC, \mathcal{N}) = \omega(C, \mathcal{N})$ for $a \in G$; (4) $\omega(C, \mathcal{N}) \leqslant n(C, K)$; (5) if S is a non-void open set with compact closure, then $\omega(\bar{S}, \mathcal{N}) \geqslant \{n(K, S)\}^{-1}$.

Property (1) follows from the obvious inequality $n(C \cup D, \mathcal{N}) \leqslant n(C, \mathcal{N}) + n(D, \mathcal{N})$ on division by $n(K, \mathcal{N})$. To verify (2) (the only remaining point whose proof uses the equicontinuity of G) we first show that there exists a set $\mathcal{N}_0$, open and non-void, such that no translate of it meets both C and D.

Let P and Q be open sets in M such that $P \parallel Q$, $P \supset C$ and $Q \supset D$. For $x \in C$ let $\tau = \tau(x)$ be such that $\mathcal{N}_\tau(x) \subset P$, and for $y \in D$ let $\tau = \tau(y)$ be such that $\mathcal{N}_\tau(y) \subset Q$. For any $\tau = \tau(z)$ let $\chi = \chi(z)$ be such that $(\mathcal{N}_\chi)^2(z) \subset \mathcal{N}_\tau(z)$, $z \in C \cup D$ (we recall that $\mathcal{N}_\mu \mathcal{N}_\nu(z)$ is defined, for any $z \in M$, to be the union of the $\mathcal{N}_\mu(w)$ as w ranges over $\mathcal{N}_\nu(z)$, and that $(\mathcal{N}_\nu)^2(z)$ is defined as $\mathcal{N}_\nu \mathcal{N}_\nu(z)$). Let $x_1, \ldots, x_m$ be a finite subset of C such that C is covered by the $\mathcal{N}_{\chi_i}(x_i)$, where $\chi_i = \chi(x_i)$, and let $y_1, \ldots, y_n$ be a finite subset of D such that D is covered by the $\mathcal{N}_{\chi^j}(y_j)$, where $\chi^j = \chi(y_j)$. Let σ be an index such that $\mathcal{N}_\sigma(p) \subset \bigcap_i \mathcal{N}_{\chi^i}(p) \cap \bigcap_j \mathcal{N}_{\chi^j}(p)$, for all $p \in M$. Observe now that if $x \in C$ and $y \in D$, then $\mathcal{N}_\sigma(x) \parallel \mathcal{N}_\sigma(y)$, for $x \in C$ implies that $x \in \mathcal{N}_{\chi_i}(x_i)$ for some i, whence $\mathcal{N}_\sigma(x) \subset \mathcal{N}_\sigma \mathcal{N}_{\chi_i}(x_i) \subset \mathcal{N}_{\chi_i} \mathcal{N}_{\chi_i}(x_i) \subset \mathcal{N}_{\tau_i}(x_i)$, where $\tau_i = \tau(x_i)$, which is in P; and similarly, $\mathcal{N}_\sigma(y) \subset Q$, so that the stated result follows from the fact that $P \parallel Q$. Finally, let π be such that $a\mathcal{N}_\pi(x) \subset \mathcal{N}_\sigma(ax)$, for $a \in G$ and $x \in M$, and ρ such that $(\mathcal{N}_\rho)^{-1}(z) \subset \mathcal{N}_\pi(z)$, $z \in M$. Note now that no translate of $\mathcal{N}_\rho(p_0)$, p_0 being an arbitrary point in M, meets both C and D, for if $x \in C, y \in D$, and $\{x, y\} \subset a\mathcal{N}_\rho(p_0)$, then $\{a^{-1}x, a^{-1}y\} \subset \mathcal{N}_\rho(p_0)$, $p_0 \in (\mathcal{N}_\rho)^{-1}(a^{-1}x) \cap (\mathcal{N}_\rho)^{-1}(a^{-1}y) \subset \mathcal{N}_\pi(a^{-1}x) \cap \mathcal{N}_\pi(a^{-1}y) \subset a^{-1}[\mathcal{N}_\sigma(x) \cap \mathcal{N}_\sigma(y)]$, and $ap_0 \in \mathcal{N}_\sigma(x) \cap \mathcal{N}_\sigma(y)$, contradicting the observation above that $\mathcal{N}_\sigma(x) \parallel \mathcal{N}_\sigma(y)$. Putting $\mathcal{N}_0$ for a non-void open

subset of $\mathcal{N}_\rho(p)$, where p is any point in M, it follows that no translate of $\mathcal{N}_0$ meets both C and D.

Now if $\mathcal{N}$ is a non-void open subset of $\mathcal{N}_0$, and if $C \cup D \subset \cup\, a_i\mathcal{N}$, where $i = 1, \ldots, n(C \cup D, \mathcal{N})$, we can write $\cup\, a_i\mathcal{N} = \bigcup_{i\varepsilon A} a_i\mathcal{N} \cup \bigcup_{i\varepsilon B} a_i\mathcal{N}$, where A is the set of i's such that $a_i\mathcal{N} \neq C$ and B is the set of i's such that $a_i\mathcal{N} \neq D$. By the way in which $\mathcal{N}_0$ was chosen, A and B are disjoint. It there are j elements in A and k elements in B it follows that $j + k = n(C \cup D, \mathcal{N})$ and it is clear that $j \geqslant n(C, \mathcal{N})$, $k \geqslant n(D, \mathcal{N})$. It results that $n(C \cup D, \mathcal{N}) \geqslant n(C, \mathcal{N}) + n(D, \mathcal{N})$, and since (1) gives the inequality in the opposite direction, we can conclude (2). Property (3) is practically obvious: if $C \subset \bigcup_i b_i\mathcal{N}$, then $aC \subset \bigcup_i ab_i\mathcal{N}$, so that $n(C, \mathcal{N}) \geqslant n(aC, \mathcal{N}) \geqslant n(a^{-1}aC, \mathcal{N}) = n(C, \mathcal{N})$. To verify (4), note that it means $n(C, \mathcal{N}) \leqslant n(C, K)\, n(K, \mathcal{N})$, which is implied by the fact that if $C \subset \bigcup_i a_iK$ and if $K \subset \bigcup_i b_j\mathcal{N}$; then $C \subset \bigcup_{i,j} a_ib_j\mathcal{N}$. Finally, (5) means that $n(K, S)\; n(S, \mathcal{N}) \geqslant n(K, \mathcal{N})$, which is valid because if $K \subset \bigcup_i a_iS$ and $S \subset \bigcup_j b_i\mathcal{N}$, then $K \subset \bigcup_{i,\,r'} a_ib_j\mathcal{N}$.

For each fixed $\mathcal{N}$, $\omega(C, \mathcal{N})$ is a real, non-negative-valued function, say ω_N, on Ω. Let Φ be the collection of all functions ϕ_N on Ω to the reals such that $0 \leqslant \phi_N(C) \leqslant n(C, K)$, $C \varepsilon \Omega$. Now Φ is set-theoretically the product, over the base set Ω, of the closed intervals $[0, n(C, K)]$, and so, by Tychonoff's theorem, is compact if regarded as a topological space in the product topology; to avoid notational complications, this space will be denoted by Φ also. Now let p_0 be any fixed point in $\mathcal{N}_0$, and put Λ_σ for the closure of $[\omega_N \mid \mathcal{N} \subset \mathcal{N}_\sigma(p_0)]$. Then the Λ_σ are compact, and have the finite intersection property: $\bigcap_{i=1}^{n} \Lambda_{\sigma i} \supset \Lambda_\sigma$ if $\mathcal{N}_\sigma \subset \bigcap_{i=1}^{n} \mathcal{N}_{\sigma i}$. Thus there is an element λ of the common part of the Λ_σ. By the definition of the product

topology, for any $\varepsilon > 0$ and any finite collection $C_1, \ldots, C_n$ of compact sets in M, there is a σ such that $\mathcal{N}_\sigma(p_0) \subset \mathcal{N}_0$, and an $\mathcal{N} \subset \mathcal{N}_0(p_0)$, such that $|\lambda(C_i) - \omega(C_i, \mathcal{N})| < \varepsilon$ $(i = 1, \ldots, n)$. It follows readily from the fact that ω satisfies (1), (2), and (3) above, that λ satisfies (1), (2), and (3) of the lemma. To see that λ is not identically zero, we observe that from (5) it results that $\lambda(\bar{S}) > \{n(K, S)\}^{-1}$ for any non-void open set S.

The next and final step in the proof of the theorem is to derive a regular measure from λ. This can be done for any λ satisfying conditions (1) and (2) of the lemma, and this fact can be used (as we show at the end of this section) to obtain the representation as an integral relative to a regular measure, for positive linear functionals on the continuous functions vanishing outside compact sets in a locally compact space. The methods used in the proof of Lemma 5.2 are due to von Neumann (unpublished lectures, 1940-41) and Kodaira[5].

LEMMA 5.2. *Let λ be a real non-negative-valued function on the compact subsets of a locally compact Hausdorff space M with the following properties (here C and D are arbitrary compact subsets of M) : (1) $\lambda(C \cap D) \leqslant \lambda(C) + \lambda(D)$; (2) if C and D have a void intersection, then $\lambda(C \cup D) = \lambda(C) + \lambda(D)$. For an open subset O of M, set $\mu_0(O) = LUB\,[C \subset O,\ C \text{ compact}]\,\lambda(C)$, and for arbitrary subsets S of M set $\mu(S) = GLB\,[O \supset S,\ O \text{ open}]\,\mu_0(O)$. Then μ is a regular measure on the σ-field generated by the compact sets and the sets $\mathcal{N}$ for which $\mu(\mathcal{N}) = 0$.*

SUBLEMMA 5.2.1. *μ coincides with μ_0 on open sets.*

If S is open, then from $S \subset S$ one concludes that $\mu(S) \leqslant \mu_0(S)$. On the other hand, if $O \supset S$ where O is open, then it is clear from the definition of μ_0 that $\mu_0(O) \geqslant \mu_0(S)$. Hence $\mu_0(S) \leqslant GLB\,[O \supset S,\ O \text{ open}]\,\mu_0(O)$, i.e. $\mu_0(S) \leqslant \mu(S)$. Combining the two inequalities yields the lemma.

SUBLEMMA 5.2.2. *If R and S are arbitrary subsets of M with $R \subset S$, then $\mu(R) \leqslant \mu(S)$.*

For if O is open and $O \supset S$, then $O \supset R$, so that $[O \mid O \supset R, O \text{ open}] \supset [O \mid \supset S, O \text{ open}]$, and since the GLB is a monotone decreasing function of the set over which it is taken, $\mu(R) = GLB[O \supset R, O \text{ open}]\, \mu(O) \leqslant GLB\,[O \supset S, O \text{ open}]\, \mu(O) = \mu(S)$.

SUBLEMMA 5.2.3. *If E is a compact subset of the union of two open subsets P and Q of M, then there exist compact subsets C and D of P and Q respectively, such that $E \subset C \cup D$.*

If $x \in E$, then either $x \in P$ or $x \in Q$. If $x \in P$, put $\mathcal{N}(x)$ for a neighbourhood of x whose closure is contained in P; if $x \in P$, put $\mathcal{N}(x)$ for a neighbourhood of x whose closure is contained in Q. Clearly, E is covered by the $\mathcal{N}(x)$ as x ranges over E, and hence there exists a finite subset X of E such that E is covered by the $\mathcal{N}(x)$ with $x \in X$. Now put C for the union of the closures of the $\mathcal{N}(x)$ as x ranges over $X \cap P$, and D for the union of the closures of the $\mathcal{N}(x)$ as x ranges over $X \cap (Q-P)$. It is easy to verify that C and D satisfy the conclusion of the sublemma.

SUBLEMMA 5.2.4. *If O_i $(i = 1, 2, \ldots, n)$ is any finite collection of open subsets of M, then $\mu(\bigcup_i O_i) \leqslant \sum_i \mu(O_i)$.*

Consider, to begin with, the case in which $n = 2$ and $\mu(O_1 \cup O_2)$ is finite. Let ϵ be an arbitrary positive number, and let C be a compact subset of $O_1 \cup O_2$ such that $\lambda(C) > \mu(O_1 \cup O_2) - \varepsilon$. Let C_i be a subset of O_i $(i = 1, 2)$ which is compact and such that $C_1 \cup C_2 \supset C$; the C_i exist, by the preceding sublemma. Now, $\mu(O_1) + \mu(O_2) \geqslant \lambda(C_1) + \lambda(C_2) \geqslant \lambda(C_1 \cup C_2) \geqslant \lambda(C) > \mu(O_1 \cup O_2) - \epsilon$. Since ϵ is arbitrary, the sublemma follows in this case.

Now if $n = 2$ and $\mu(O_1 \cup O_2)$ is infinite, for any real there exists a subset C of $O_1 \cup O_2$ such that $\lambda(C) > \alpha$. The inequalities just obtained show that there exist compact subsets C_i of O_i $(i = 1, 2)$ such that $\lambda(C_1) + \lambda(C_2) > \alpha$, and this implies that $\mu(O_1) + \mu(O_2) > \alpha$, and hence that $\mu(O_1) + \mu(O_2)$ is infinite. For arbitrary $n > 2$ we have $\mu(\bigcup_{i=1}^{n} O_i) =$

$\mu(\bigcup_{i<n} O_i \cup O_n) \leqslant \mu(\bigcup_{i<n} O_i)+\mu(O_n)$, and the sublemma follows by induction.

SUBLEMMA 5.2.5. *For any sequence* $\{ O_n \}$ *of open subsets of* M, $\mu(\bigcup_n O_n) \leqslant \Sigma_n \mu(O_n)$.

Suppose first that $\mu(\bigcup_n O_n)$ is finite. Let ε be an arbitrary positive number, and let C be a compact subset of $\bigcup_n O_n$ such that $\lambda(C) > \mu(\bigcup_n O_n)-\varepsilon$. Since C is compact, it is covered by a finite number of the O_n, say by the O_n with $n \varepsilon A$, where A is a finite set of indices. Then $\mu(\bigcup_n O_n-)\varepsilon \leqslant \lambda(C) \leqslant \mu(\bigcup_{n\varepsilon A} O_n)$ (by the definition of μ) $\leqslant$ $\sum_{n\varepsilon A} \mu(O_n)$ (by the previous sublemma) $\leqslant \sum_n \mu(O_n)$. Since ε is arbitrary, the lemma follows in this case.

Now if $\mu(\bigcup_n O_n)$ is infinite, for any real α there exists a compact subset C of $\bigcup_n O_n$ such that $\lambda(C) > \alpha$. It follows from the preceding inequalities that $\sum_n \mu(O_n) > \alpha$, and hence that $\sum_n \mu(O_n)$ is infinite.

SUBLEMMA 5.2.6. *For any sequence* $\{ S_n \}$ *of subsets of* M, $\mu(\bigcup_n S_n) \leqslant \sum_n \mu(S_n)$.

It is trivial to conclude from this lemma that μ is a Carathéodory outer measure on M.

To prove the sublemma, it is sufficient to consider the case in which $\mu(S_n)$ is finite for every n, as otherwise it follows immediately from Sublemma 5.2.2. Let ε be an arbitrary positive number, and let O_n be an open set containing S_n and such that $\mu(S_n) > \mu(O_n)-\varepsilon 2^{-n}$ $(n = 1, 2,\ldots,)$. Then $\bigcup_n S_n \subset \bigcup_n O_n$ and $\mu(\bigcup_n S_n) \leqslant \mu(\bigcup_n O_n) \leqslant \sum_n \mu(O_n)$ $< \sum_n (\mu(S_n)+\varepsilon/2^n) = \sum_n \mu(S_n)+\varepsilon$, and ε being arbitrary, the sublemma follows.

SUBLEMMA 5.2.7. *If P and Q are disjoint open subsets of G, then* $\mu(P \cup Q) = \mu(P)+\mu(Q)$.

It is clearly sufficient to consider the case in which both $\mu(P)$ and $\mu(Q)$ are finite. Let ε be an arbitrary positive number and let C and D be compact subsets of P and Q respectively, such that the values of μ and λ on the open and respective compact sets differ by less than ε. Then C and D are disjoint and $\mu(P)+\mu(Q)>\{\lambda(C)+\varepsilon\}+\{\lambda(D)+\varepsilon\}=\lambda(C\cup D)+2\varepsilon$ (since λ is additive on disjoint compact sets) $<\mu(P\cup Q)+2\varepsilon$, and the sublemma follows from Sublemma 5.2.4 and the circumstance that ε is arbitrary.

SUBLEMMA 5.2.8. *If P and Q are open subsets of M, then $\mu(P)=\mu(P\cap Q)+\mu(Q-P)$.*

Let ε be an arbitrary positive number, let C be a compact subset of $P\cap Q$ such that $\lambda(C)>\mu(P\cap Q)$, and let R be open and such that $P\cap Q\supset\bar{R}\supset R\supset C$. Then since $Q-P\subset Q-\bar{R}$, we have $\mu(P\cap Q)+\mu(Q-P)\leqslant\mu(P\cap Q)+\mu(Q-\bar{R})<\lambda(C)+\varepsilon+\mu(Q-\bar{R})\leqslant\mu(R)+\varepsilon+\mu(Q-\bar{R})=\mu\{(Q-\bar{R})\cup R\}+\varepsilon$ (for $Q-\bar{R}$ and R are disjoint open subsets and Sublemma 5.2.9 applies) $<\mu(Q)+\varepsilon$. Hence $\mu(P\cap Q)+\mu(Q-P)\leqslant\mu(Q)$, and since Sublemma 5.2.6 implies the inequality in the other direction, the sublemma follows.

SUBLEMMA 5.2.9. *Open subsets of M are μ-measurable; that is, for any open subset O and arbitrary subset S, $\mu(S)=\mu(O\cap S)+\mu(S-O)$.*

As before, it is sufficient to consider the case in which $\mu(S)$ is finite. Let ε be an arbitrary positive number, and let P be an open set containing S and such that $\mu(S)>\mu(P)-\varepsilon$. Then $O\cap S\subset O\cap P$ and $S-O\subset P-O$, so that $\mu(O\cap S)+\mu(S-O)\leqslant\mu(O\cap P)+\mu(P-O)=\mu(P)$ (by Sublemma 5.2.8) $<\mu(S)+\varepsilon$. Hence $\mu(O\cup S)+\mu(S-O)\leqslant\mu(S)$, and since the opposite inequality follows from Sublemma 5.2.6, the proof of the sublemma is complete.

SUBLEMMA 5.2.10. *For any open subset O of M, $\mu(O)=$ LUB* $[C\subset O, C$ compact$]$ $\mu(C)$.

It is clear from the definition of $\mu(O)$ that $\mu(O) \geqslant \lambda(C)$. Hence $\mu(C) = GLB$ $[O \supset C, O$ open$]$ $\mu(O) \geqslant \lambda(C)$. It follows that LUB $[O \supset C, C$ compact$]$ $\mu(C) \geqslant \mu(O)$, and it is clear from Sublemma 5.2.2. that the inequality is also valid in the opposite direction.

Completion of proof of Lemma 5.2.2. Clearly, μ is a Caratheodory outer measure on M, and the open sets are μ-measurable. Hence μ is countably additive on the σ field determined by the open subsets and the sets of μ-measure zero, and hence also on the subfield F of σ-finite sets in this field. Now if S is a σ-finite μ-measurable set, then S is contained in a σ-finite open set, for S is then a countable union of measurable sets of finite measure, each of which is contained in an open set of finite measure. Combining this fact with Sublemmas 5.2.1 and 5.2.10, it results that μ is regular on F.

Completion of proof of theorem. It is clear that the measure μ of Lemma 5.2.2 which is constructed from the functional λ of Lemma 5.2.1 is a regular measure on G which is invariant under G and not identically zero.

We note that Lemma 5.2.2 can be used to prove that a positive linear functional on the space of continuous functions vanishing outside compact sets in a locally compact space, can be represented as an integral relative to a regular measure. Although this result is not needed in the following, it is of some general interest, and we conclude this section by deriving it.

Theorem 6. *Let X be the collection of all real-valued continuous functions on M which vanish outside compact sets, and let Λ be a linear functional on X which is non-negative on non-negative-valued functions. Then there exists a unique regular measure μ on M such that for $f \in X$, $\Lambda(f) = \int_M f(p)\, d\mu\,(p)$.*

For an arbitrary compact subset C of M, define $\lambda(C)$ to be the greatest lower bound of $\Lambda(f)$, as f ranges over the class $\Delta(C)$ of those elements of X which have values between zero and one, and which are one on C. To see

that condition (1) of Lemma 5.2 is satisfied, let ε be a positive number and let f and g be elements of $\Delta(C)$ and $\Delta(D)$ respectively such that $\wedge(f) < \lambda(C)+\varepsilon$, $\wedge(g) < \lambda(D)+\varepsilon$. Let h be an element of $\Delta(C \cup D)$ such that $h(p) \leqslant f(p)+g(p)$, for $p \in M$, e.g. let k be any element of $\Delta(C \cup D)$ and set $h(p) = \min[k(p), f(p)+g(p)]$. Then $\wedge(h) \leqslant \wedge(f)+\wedge(g)$, whence $\lambda(C \cup D) < (\lambda(C)+\varepsilon)+(\lambda(D)+\varepsilon)$, and (1) follows. It is straightforward to verify that (2) is satisfied.

Now let μ be the measure corresponding to λ by Lemma 5.2. It remains to verify that if $f \in X$, then $\wedge(f) = \int_M f(p)\, d\mu(p)$. We show first $\lambda(C) = \mu(C)$ for any compact set C. Now if P is open and contains C, then $\mu(P) \geqslant \lambda(C)$, and since $\mu(C)$ is the GLB of such $\mu(P)$, $\mu(C) \geqslant \lambda(C)$. Let $\{f_n\}$ be a sequence of elements of $\Delta(C)$ such that $\wedge(f_n) \rightarrow \lambda(C)$, and let $\{P_n\}$ be a sequence of open subsets of M such that $P_n \supset P_{n+1} \supset C$ and $\mu(P_n) \rightarrow \mu(C)$ (as $n \rightarrow \infty$). Then if g_n is in $\Delta(C)$ and vanishes outside P_n, $g_n(p) \rightarrow k(p)$ n.e., where k is the characteristic function of C. Putting $h_n(p) = \min[(g_n(p), f_n(p)]$ we have $\wedge(h_n) \rightarrow \lambda(C)$ and $h_n(p) \rightarrow k(p)$ n.e. Now let $C_n = [p \mid f_n(p) \geqslant 1-n^{-1}]$, and $Q_n = [p \mid f_n(p) > 1-n^{-1}]$. Clearly, $C_n \supset Q_n \supset C$. Also, $\mu(Q_n)$ is the LUB of $\lambda(D)$, as D ranges over the compact subsets of Q_n, and since it is plain that λ is monotone increasing, it follows that $\mu(Q_n) \leqslant \lambda(C_n)$. But $(1-n^{-1})^{-1} h_n$ majorizes an element of $\Delta(C_n)$, so that $\lambda(C_n) < \wedge(h_n)(1-n^{-1})^{-1}$, and $\lim\inf \lambda(C_n) < \lambda(C)$. Thus $\lim\inf \mu(Q_n) \leqslant (C)$, from which it results that $\mu(C) \leqslant \lambda(C)$.

To complete the proof (apart from uniqueness) it clearly suffices to show that if $f \in X$ and if f has non-negative values, then $\wedge(f) = \int_M f(p)\, d\mu(p)$. Let f be such a function, with values in the interval $[0, \gamma-1]$, say. Since the measure of a compact set is finite, there are at most a countable set of α's for which $f^{-1}(\alpha)$ has positive measure. Now for each positive integer n, let $\alpha_{i,n}$ $(i = 0, 1, \ldots, n)$ be such that: (*a*) $0 = \alpha_{0,n}$, $\gamma = \alpha_{n,n}$, and $\alpha_{i,n} < \alpha_{i+1,n}$ for all $i < n$; (*b*) $f^{-1}(\alpha_{i,n})$ has measure zero

if $i \geqslant 1$; (c) $LUB_{i<n} |\alpha_{i,n} - \alpha_{i+1,n}| \to 0$ as $n \to \infty$. Next, for $i \geqslant 1$, choose $\varepsilon_{i,n} > 0$ so that $\mu(S_{i,n}) - \mu(T_{i,n}) < n^{-2}$, where $S_{i,n} = [p \mid \alpha_{i,n} \leqslant f(p) \leqslant \alpha_{i+1,n}]$ and $T_{i,n} = [p \mid \alpha_{i,n} + \varepsilon_{i,n} \leqslant f(p) \leqslant \alpha_{i+1,n} - \varepsilon_{i,n}]$. Put $f_{i,n}$ $(i \geqslant 1)$ for an element of $\Delta(T_{i,n})$ which vanishes outside $S_{i,n}$. Note that $f(p) \geqslant f_n(p)$ for $p \in M$, where f_n is defined by the equation $f_n(p) = \sum_{i=1}^{n-1} \alpha_{i,n} f_{i,n}(p)$. It follows that $\int_M f(p)\, d\mu(p) =$

$$\lim_{n\to\infty} \sum_{i=1}^{n-1} \alpha_{i,n}\, \mu(S_i, n) = \lim_{n\to\infty} \sum_{i=1}^{n-1} \alpha_{i,n}\, \mu(T_i, n) = \lim_{n\to\infty} \sum_{i=1}^{n-1} \alpha_{i,n}$$

$$\lambda(T_i, n) \leqslant \liminf \sum_{i=1}^{n-1} \alpha_{i,n} \Lambda(f_i, n) = \liminf \Lambda(f_n) \leqslant \Lambda(f).$$

Now set $\Lambda'(g) = \Lambda(g) - \int_M g(p)\, d\mu(p)$, $g \in X$. Then Λ' is a linear functional on X which is non-negative on non-negative-valued functions. Let λ' be the corresponding functional on the compact subsets of M and μ' the corresponding regular measure. Let $\{g_n\}$, $\{h_n\}$, and $\{k_n\}$ be sequences of elements of $\Delta(C)$ with the following properties: $\Lambda'(g_n) \to \lambda'(C)$; $\Lambda(h_n) \to \lambda(C)$; $\int_M k_n(x)\, d\mu(x) \to \mu(C)$. Putting r_n for the element of $\Delta(C)$ defined by the equation $r_n(p) = \min(g_n(p), h_n(p), k_n(p))$, it follows that $\lambda'(C) = \lim_{n\to\infty} \Lambda'(r_n) = \lim_{n\to\infty} (\Lambda(r_n) - \int_M r_n(p)\, d\mu(p)) = \lambda(C) - \mu(C) = 0$. Thus $\lambda'(C) = 0$ for all C, and hence $\Lambda' = 0$.

It is easy to verify, using sequences of functions similar to those used above, that the measure μ is unique.

7. Uniqueness of invariant measures. A number of short proofs of the uniqueness of Haar measure depend on the Fubini theorem, which becomes applicable as soon as appropriate measurability results, such as those of Sections 4 and 5 are available. The first such proof was given by von Neumann(6), and among the shortest such proofs are those of Kakutani(4) and Weil [9, p. 148-9].

But for extension to the case of a group of transformations of a space, a quite different kind of proof, due to Weil [9, p. 37-38], appears to be more suitable. The following theorem constitutes such an extension, and its proof is similar to this latter proof.

THEOREM 7. *Let G be an equicontinuous group of unimorphisms of a uniformly locally compact space M. Then the regular measures on M which are invariant under G are all proportional to each other, if and only if there is a point in M whose orbit is dense.*

First we show that the existence of a dense orbit is necessary for uniqueness. The following proof shows incidentally that if one orbit is dense, then all are. Suppose now that p_0 is a point of M whose orbit is not dense, and let L be the closure of its orbit. Then in the relative uniform structures, L and $M-L$ are uniformly locally compact, and G leaves both of these relative spaces invariant, and acts equicontinuously on them. Hence, by Theorem 5, there exist non-zero regular invariant measures λ and μ, on L and $M-L$ respectively. Now we define measures λ' and μ' on M as follows: if E is any set in the σ-field generated by the compact subsets of M, then $\lambda'(E) = \lambda(E \cap L)$ and $\mu'(E) = \mu(E \cap (M-L))$. Then it is easy to see that, except possibly for completeness, λ' and μ' are regular, and it is clear that they are invariant. For any real non-negative α and β, the completion of $\alpha\lambda'+\beta\mu'$ is an invariant regular measure on M, and it is straightforward to verify that the collection of all such measures cannot consist solely of multiples of some one measure.

Next we show the converse; we assume that there exists a point p_0 in M whose orbit is dense, and prove that any two regular invariant measures are proportional. Now any regular measure μ determines a linear functional on X (X being as in Theorem 6), by the equation $\wedge(f) = \int_M f(p)\, d\mu(p)$, and μ is determined by the correspond-

ing functional $\wedge$. Hence it is enough to show that the ratio $\wedge(f)/\wedge(f_0)$, where f_0 is any fixed element of X for which $\wedge(f_0) \neq 0$, and $\wedge$ is the functional on X determined by a regular invariant measure on M, is completely determined by the fashion in which G acts on M.

For any two elements f and g of X^+, where X^+ is the collection of all non-negative elements of X, let $\wedge(f, g)$ be defined as follows: $\wedge(f, g)$ is the GLB of $\sum_{i=1}^{n} \alpha_i$, where the α_i are any real non-negative numbers such that $f(p) \leqslant \sum_{i=1}^{n} \alpha_i g(a_i^{-1}p)$, for some $\{a_i\} \subset G$, provided such constants exist; otherwise, $\wedge(f, g)$ is to be undefined. It is evident that $\wedge(f, g)$ is determined by the action of G on M. We note that if $\wedge(f, h)$ and $\wedge(h, g)$ are both defined, then so is $\wedge(f, g)$, and $\wedge(f, g) \leqslant \wedge(f, h)$ $\wedge(h, g)$; for if $f(p) \leqslant \Sigma\alpha_i h(a_i^{-1}p)$ and $h(p) \leqslant \Sigma\beta_i g(b_j^{-1}p)$, then $f(p) \leqslant \Sigma\alpha_i\beta_i h((a_i b_j)^{-1})$, and $\Sigma\alpha_i\beta\alpha_i = (\Sigma\alpha_i)(\Sigma\beta_i)$.

Now let f be in X^+, and let ϵ be any positive number. Then since f is uniformly continuous, there exists an index σ such that if $p \in N_\sigma(q)$, then $|f(p) - f(q)| < \epsilon$. Let g be in X^+, and such that g does not vanish at p_0 but does vanish outside $N_\rho(p_0)$, where ρ is chosen so that $aN_\rho \subset N_\sigma(p)(ap)$, $a \in G$ and $p \in M$. Let μ be the invariant measure to which $\wedge$ corresponds, and set $h(a) = \int_M f(p)\, g(a^{-1}p)\, d\mu(p)$. Since $g(a^{-1}p) = 0$ unless $a^{-1}p \in N_\rho(p_0)$, i e. $p \in aN_\rho(p_0) \subset N_\sigma(ap_0)$, and then $|f(p) - f(ap_0)| < \epsilon$, so that $f(p) > f(ap_0) - \epsilon$, we have

$$h(a) \geqslant \{f(a p_0) - \varepsilon\} \int g(a^{-1}p)\, d\mu(P) = \{f(ap_0) - \varepsilon\} \wedge(g).$$

Now let δ be an arbitrary positive number. Since g is uniformly continuous, there exists an index ω such that if $p \in N_\omega(q)$, then $|g(p) - g(q)| < \delta$. Let τ be such that $aN_\tau(p) \subset N_\omega(ap)$, for $a \in G$ and $p \in M$, and let C be a compact

set outside of which f vanishes. Now by the first part of the proof of Lemma 5. 1 (specifically, the proof that G is weakly transitive on P), the only invariant non-void open set in M is M itself. Hence, C is covered by the translates of $\mathcal{N}_\tau(p_0)$; let $\{ a_i; i = 1, \ldots, n \}$ be a finite set of elements of G such that the $a_i\mathcal{N}_\tau(p_0)$ cover C. Let $\{ k_i \ (i = 1, \ldots, n) \}$ be an element of X^+ which vanishes outside of $a_i\mathcal{N}_\tau(p_0)$, and is such that for $p \varepsilon C, \Sigma_i k_i(p) = 1$; it is well known, and not difficult to verify that such functions exist. Then $h(a) = \sum_i \int_M f(p)\, k_i(p)\, g(a^{-1}p)\, d\mu(p)$, and since k_i vanishes outside $a_i\mathcal{N}_\tau(p_0)$, when $p \varepsilon a_i \mathcal{N}_\tau(p_0)$, $a^{-1}p \varepsilon a^{-1}a_i\mathcal{N}(p_0) \subset \mathcal{N}_\omega(a^{-1}a_ip_0)$, so that $g(a^{-1}p) < g(a^{-1}a_ip_0) + \delta$, and we have $h(a) \leqslant \Sigma_i \wedge fk_i) \ \{ g(a^{-1}a_ip_0) + \delta \}$.

Now assume that g satisfies the following condition: $g(a^{-1}p_0) = g(ap_0), a \varepsilon G$. Then $h(a) \leqslant \Sigma_i \wedge (fk_i) \ \{ g(a_i^{-1}ap_0) + \delta \}$, and $f(ap_0) - \epsilon \leqslant \Sigma \alpha_i \{ g(a_i^{-1}ap_0) + \delta \}$, where $\alpha_i = <(fk_i) / \wedge(g)$. (It should be observed that $\wedge(g) \neq 0$ unless $\mu = 0$, for $g(p_0) \neq 0$, and for any neighborhood $\mathcal{N}$ of p_0, $\mu(\mathcal{N}) \neq 0$, because the translates of $\mathcal{N}$ cover M; and we clearly may and do, assume that $\mu \neq 0$). Since the orbit of P_0 is dense, and since f and g are continuous, we have $f(p) - \epsilon \leqslant \Sigma_i \alpha_i \{ g(a_i^{-1}p) + \delta \}$. Putting m for an element of X^+ which is one at every point of C, it results that $f(p) \leqslant (\varepsilon + \delta \Sigma \alpha_i)\, m(p) + \Sigma \alpha_i g(a_i^{-1}p)$. Now $\wedge(s, g)$ exists for every $s \epsilon X^+$, for if s vanishes outside the compact set D and is bounded by α, then taking any b_j (j ranging over a finite set) such that the $b_j\mathcal{N}(p_0)$ cover D, where $\mathcal{N}(p_0)$ is a neighbourhood of p_0 on which g is bounded from below by $\gamma > 0$, we have $s(p) \leqslant \alpha\gamma^{-1} \Sigma_i g(b_j^{-1}p)$. Hence, noting that $\Sigma \alpha_i = \wedge(f) / \wedge(g) \leqslant (\epsilon + \delta \Sigma \alpha_i) \ \wedge(m, g) + \wedge(f) / \wedge(g)$, and that δ is arbitrary, it follows that $\wedge(f, g) \leqslant \varepsilon \wedge (m, g) + \wedge(f) / \wedge(g)$. On the other hand, if $f(p) \leqslant \Sigma_i \alpha_i g(a_i^{-1}p)$, then plainly $\wedge(f) \leqslant (\Sigma \alpha_i) \wedge(g)$, which implies that $\wedge(f, g) \geqslant \wedge(f) / \wedge(g)$.

To conclude the proof we need chiefly to show that functions g with the properties used above do actually exist (which of course is clear in case that $G = M$ and acts on itself by left translation). In this connection we use the fact[10] that a uniform structure can always be taken to be defined by a family of pseudometrics,—i.e. for any structure there is an equivalent one in which the neighbourhood families have the form $N_\sigma(p) = [q \mid d_\sigma(p, q) < 1]$, where d_σ is a function on pairs of points of M satisfying the usual axioms for a metric, except that its vanishing does not imply that the points are identical. We assume now that N_σ has this form, and observe that there is no loss of generality in assuming further that d_σ is bounded, since otherwise d_σ could be replaced by $\phi(d_\sigma)$, where ϕ is a bounded convex function such that $\phi(\alpha) = \alpha$ for $0 \leqslant \alpha \leqslant 1$. Next we observe that if ρ is such that $aN_\rho(p) \subset N_\sigma(ap)$ for $a \in G$ and $p \in M$ then $LUB_{a \in G}\, d_\rho(ap, aq)$ exists and defines a function $d'_\rho(p, q)$, which is a group-invariant pseudometric, and such that $N'_\rho(p) \subset N_\sigma(p)$, for $p \in M$, where $N'_\rho(p) = [q \mid d'_\rho(p, q) < 1]$. Now setting $g(p) = \psi[d'_\rho(p, p_0)]$, where $\psi(\alpha) = 1 - \alpha$ for $0 \leqslant \alpha \leqslant 1$ and $= 0$ for $\alpha \geqslant 1$, it is clear that g is continuous, vanishes outside of $N_\sigma(p_0)$, and $g(ap_0) = \psi[d'(ap_0, p_0)] = \psi[d'(p_0, a^{-1}p_0)] = g(a^{-1}p_0)$.

The inequalities above can now be put in the form $\wedge(g) \ \wedge(f, g) - \varepsilon \wedge(g) \ \wedge(m, g) \leqslant \wedge(f) \leqslant \wedge(g) \ \wedge(f, g)$. Putting f_0 for any fixed f which does not vanish at p_0 and is such that $\wedge(f_0) \neq 0$, and m_0 for a corresponding m, it results that

$$\frac{\wedge(f, g) - \epsilon \wedge(m, g)}{\wedge(f_0, g)} \leqslant \frac{\wedge(f)}{\wedge(f_0)} \leqslant \frac{\wedge(f, g)}{\wedge(f_0, g) - \epsilon \wedge(m_0\, g)},$$

and in order to complete the proof that $\wedge(f)/\wedge(f_0)$ is the limit with respect to σ of $\wedge(f, g)/\wedge(f_0)$ (and thereby determined by the action of G on M, p_0 being an arbitrary point with a dense orbit), it is easily seen to be sufficient to show that $\wedge(m, g)/\wedge(f_0, g)$ and $\wedge(m_0, g)/\wedge(f_0, g)$ are bounded as functions of g. Now $\wedge(k, f_0)$

exists for any $k \in X^+$, since $f_0(p_0) \neq 0$. Applying the inequality obtained at the beginning of the proof, we obtain that $\wedge(m_0, g)/\wedge(f_0, g) \leqslant \wedge(m_0, f_0)$ and $\wedge(m, g)/\wedge(f_0, g) \leqslant \wedge(m, f_0)$.

COR. 7.1. *In the case of a locally compact group with a left-invariant metric, Haar measure is invariant under all isometries of the group onto itself.*

This is clear from the observation that the collection of all isometries of the group onto itself is an equicontinuous group which includes all left translations.

COR. 7.2. *An equicontinuous group of unimorphisms of a uniformly locally compact space which leaves invariant a regular measure is ergodic, if there is a point in the space whose orbit is dense.*

This follows from the uniqueness of the invarivant measure just as in the case of a group acting on itself (see e. g.[(8)]), and we omit details.

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ON A TAUBERIAN THEOREM CONNECTED WITH THE NEW PROOF OF THE PRIME NUMBER THEOREM

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In our new and elementary proof of the prime number theorem, Selberg and I[1] prove the following Tauberian theorem:

Let $1 < p_1 < p_2 < \ldots$ be an infinite sequence of real numbers which satisfies

$$I. \quad \sum_{p_i \leq x} (\log p_i)^2 + \sum_{p_i p_j \leq x} \log p_i \log p_j = 2x \log x + o(x \log x)$$

$$II. \quad \sum_{p_i \leq x} \frac{\log p_i}{p_i} = [1+o(1)] \log x$$

$$III. \quad \vartheta(x) = \sum_{p_i \leq x} \log p_i > cx.$$

Then

$$\lim_{x \to \infty} \vartheta(x)/x = 1. \tag{1}$$

I is the fundamental asymptotic formula of Selberg[2] for which he obtained an ingenious and elementary proof, and which was the starting point of our investigations. *II* is due to Mertens and *III* to Tchebichef, both of course have well-known elementary proofs.

It might be of interest to investigate whether (1) can be deduced from *I* alone. This indeed turns out to be the case if we use *I* with the stronger error term $O(x)$ (which indeed was proved by Selberg[2]). Thus we shall prove the following

THEOREM 1. *Let* $1 < p_1 < p_2 < \dots$ *Assume that*

$$I'. \quad \sum_{p_i \leq x} (\log p_i)^2 + \sum_{p_i p_j \leq x} \log p_i \log p_j = 2x \log x + O(x).$$

Then (1) *holds.*

We shall prove Theorem 1 by showing that I' implies

$$II'. \quad \sum_{p_i \leq x} \frac{\log p_i}{p_i} = \log x + O(1).$$

II and III are easy consequences of II', (thus (1) follows from our work with Selberg[1]). This is clear for II and easy for III, for if III does not hold, we have $\vartheta(x) < \epsilon x$ with $\epsilon < 1$ for suitable x. Then we evidently have

$$\sum_{x\sqrt{\epsilon}}^{x} \frac{\log p_i}{p_i} < \frac{\vartheta(x)}{\epsilon^{1/2} x} < \epsilon^{1/2},$$

thus II' cannot hold[3].

Our principal tool in proving Theorem 1 will be the following Tauberian theorem which is of interest by itself:

THEOREM 2. *Let* $a_k \geq 0$, *put* $s_m = \sum_{k=1}^{m} a_k$. *Assume that*

$$S(n) = \sum_{k=1}^{n} a_k(s_{n-k} + k) = n^2 + O(n). \tag{2}$$

Then

$$s_n = n + O(1).$$

First we shall show that I' implies

$$I'': \quad \sum_{p_i \leq x} \frac{(\log p_i)^2}{p_i} + \sum_{p_i p_j \leq x} \frac{\log p_i \log p_j}{p_i p_j} = (\log x)^2 + O(\log x).$$ [4]

We use partial summation. Put

$$D(u) = \sum_{p_i \leq u} (\log p_i)^2 + \sum_{p_i p_j \leq u} \log p_i \log p_j = 2u \log u + O(u).$$

We evidently have

$$A(x) = \sum_{p_i < x} \frac{(\log p_i)^2}{p_i} + \sum_{p_i p_j \leqslant x} \frac{\log p_i \log p_j}{p_i p_j}$$

$$= \sum_{p_i \leqslant x} \frac{(\log p_i)^2}{[p_i]} + \sum_{p_i p_j \leqslant x} \frac{\log p_i \log p_j}{[p_i p_j]}$$

$$+ O\Big(\sum_{p_i \leqslant x} \frac{(\log p_i)^2}{p_i} + \sum_{p_i p_j \leqslant x} \frac{\log p_i \log p_j}{(p_i p_j)^2} \Big)$$

$$= \sum_{p_i \leqslant x} \frac{(\log p_i)^2}{[p_i]} + \sum_{p_i p_j \leqslant x} \frac{\log p_i \log p_j}{[p_i p_j]} + O(1)$$

(since $D(u) = O(u \log u)$). Thus

$$A(x) = \sum_{u=1}^{x} \frac{D(u+1) - D(u)}{u} + O(1)$$

$$= \sum_{u=2}^{x-1} \frac{D(u)}{u(u-1)} + \frac{D(x)}{x} + O(1) = (\log x)^2 + O(\log x),$$

which proves II'.

Put now

$$e^n = x, \; a_k = \sum_{e^{k-1}}^{e^k} \frac{\log p_i}{p_i} < \frac{\vartheta(e^k)}{e^{k-1}} < c.$$

Then we obtain from I'' by a simple computation

$$\sum_{k=1}^{n} k a_k + O\Big(\sum_{k=1}^{n} a_k \Big) + \sum_{k=1}^{n} a_k (a_1 + a_2 + \ldots + a_{n-k})$$

$$+ O\Big(\sum_{k=1}^{n} a_k a_{n-k+1} \Big) = n^2 + O(n). \quad (3)$$

We obtain (3) from I'' by putting $\dfrac{(\log p_i)^2}{p_i} = \dfrac{\log p_i \log p_i}{p_i}$ $= k \dfrac{\log p_i}{p_i} + O\left(\dfrac{\log p_i}{p_i} \right)$, for $e^{k-1} \leqslant p_i < e^k$ and by remarking that if $e^{k-1} \leqslant p_i < e^k$ then in $p_i p_j \leqslant e^n$, $e^{n-k} \leqslant p_j < e^{n-k+1}$, also $a_k < c$.

Thus finally we obtain from (3)

$\sum_{k=1}^{n} a_k (s_{n-k}+k) = n^2+O(n), 0 \leqslant a_k$ ($a_k < c$ is not needed any more). Hence from Theorem 2

$$s_n = n+O(1),$$

or in other words II' holds, which proves Theorem 1.

Thus we only have to prove Theorem 2. The difficulty is caused by the sharpness of the error term. First to illustrate our method we prove that

$$s_n = n+O(\log n). \tag{4}$$

In the proof we shall use some of the ideas used in our proof with Selberg(1). We shall prove (4) in stages. The first step is to prove that

$$s_n = n+o(n). \tag{5}$$

Apply (2) for n and $n+1$ and subtract. Then we obtain

$$(n+1)\, a_{n+1} < 2n+O(n) \text{ (since } a_k \geqslant 0),$$

or

$$a_n \leqslant c. \tag{6}$$

Put

$$A = \overline{\lim}\, \frac{s_n}{n},\ a = \underline{\lim}\, \frac{s_n}{n}.$$

We obtain from (2) by a very simple argument that $a \leqslant 1, A \geqslant 1$. In fact if $a \leqslant 1$ we have $s_{n-k}+k \geqslant n+o(n)$. Thus from (2)

$$n^2+O(n) = \sum_{k=1}^{n} a_k (s_{n-k}+k) \geqslant (n+o(n)) \sum_{k=1}^{n} a_k$$

or $A \leqslant 1$, i.e. $a = A = 1$. Hence we can assume $a < 1$, and similarly $A > 1$.

(6) implies $A < \infty$. Choose n so that $s_n = A.n+o(n)$. We clearly can assume $a < A$ (otherwise (5) holds and there is nothing to prove). For $k \leqslant \epsilon n$ (ϵ small but fixed) we have by (6)

$$s_{n-k}+k \geqslant s_n - k\, c+k \geqslant An - \epsilon\, c\, n - o(n) > (a+\delta)\, n, \tag{7}$$

where $\delta = \delta(\epsilon) > 0$ is a fixed number. For $k > \epsilon n$ we have

$$s_{n-k}+k \geqslant a\ (n-k)+k-o(n) = an+k\,(1-a)-o(n)$$
$$> an+\epsilon n\ (1-a)-o(n) > (a+\delta)\ n \tag{8}$$

Hence from (2), (7) and (8) we have

$$n^2+O\ (n) = \sum_{k=1}^{n} a_k\,(s_{n-k}+k) \geqslant (a+\delta)\,n.\ \sum_{k=1}^{n} a_k$$
$$= A\,a\,n^2+A\,\delta\,n^2+o(n^2),$$

or $A\,a<1$. By choosing n so that $s_n = an+o(n)$, we obtain similarly $A\,a>1$, an evident contradiction, which proves that $A=a=1$, hence (5) is proved.

Next we prove

$$s_n = n+o(n^\epsilon). \tag{9}$$

Put

$$s_n = n+B_n, B_n = o(n), \text{by } (5).$$

We can assume that $\overline{\lim}\ B_n = \infty$ (otherwise there is nothing to prove). Denote

$$\bar{B}_n = \max_{m \leqslant n} |B_m|.$$

First we assume that

$$\overline{\lim_{n=\infty}}\ \bar{B}_n/\bar{B}_{n/2} > 1+c. \tag{10}$$

Choose n so that

$$\bar{B}_n/\bar{B}_{n/2} > 1+c, \tag{11}$$

without loss of generality we my assume that $|B_n| = \bar{B}_n$. For if not, we have for some m, $n/2<m\leqslant n$, $|\bar{B}_m| = \bar{B}_n$ and clearly $\bar{B}_m/\bar{B}_{m/2} \geqslant \bar{B}_n/\bar{B}_{n/2} > 1+c$. Further we can assume without loss of generality (as will be clear from our proof) that $B_n = \bar{B}_n$. We have by (11) and the definition of $\bar{B}_n$, $\bar{B}_{n/2}$ for every k

$$s_{n-k}+k \geqslant n-k-B_n+k = n-B_n \tag{12}$$

and for $k>n/2$

$$s_{n-k}+k \geqslant n-k-\bar{B}_{n/2}+k > n-k-B_n/(1+c)+k > n-B_n+c_1\,B_n. \tag{13}$$

Thus from (2), (12) and (13)

$$n^2+O(n)=\sum_{k=1}^{n} a_k(s_{n-k}+k) > \sum_{k=1}^{n} a_k(n-B_n)+c_1 B_n \sum_{k=n/2}^{n} a_k.$$

Now we have from (5)

$$\sum_{k=1}^{n} a_k = n+B_n, \quad \sum_{n/2}^{n} a_k = \frac{n}{2}+o(n) > \frac{n}{4}.$$

Thus finally

$$n^2+O(n) > n^2-B_n^2+c_1 B_n\, n$$

which is false since $\bar{B}_n\,(=B_n)\to\infty$ and $B_n = o(n)$.

Thus we can assume that

$$\lim_{n=\infty} \bar{B}_n/\bar{B}_{n/2} = 1. \tag{14}$$

But then we immediately obtain by iterating (14) that $B_n = o(n^{\varepsilon})$, which proves (9).

Now we are ready to prove (4). Assume that (4) does not hold. Then we clearly can assume that

$$\overline{\lim}\,(\bar{B}_n - \bar{B}_{n/2}) = \infty.$$

Choose n so that $\bar{B}_n - \bar{B}_{n/2} > C$. As before we can assume that $\bar{B}_n = B_n$. As in (12) and (13) we have

$s_{n-k}+k \geqslant n - B_n$ for all k; $s_{n-k}+k > n - B_n + C$ for $k > n/2$.

Thus we have as before

$$n^2+O(n) = \sum_{k=1}^{n} a_k(s_{n-k}+k) > \sum_{k=1}^{n} a_k(n-B_n)$$

$$+C\sum_{n/2}^{n} a_k > n^2-B_n^2+\frac{C}{4}n,$$

which is false since $B_n = o(n^{\varepsilon})$ and C can be chosen as large as we please. This contradiction proves (4).

Unfortunately it seems that one cannot get a stronger result by this method.

Now we show that to prove Theorem 2 it will suffice to assume that for sufficiently large k, $a_k < 2+\epsilon$, where ϵ is an arbitrarily small but fixed positive number.

Assume then that Theorem 2 is proved in the case $a_k < 2+\epsilon$. Then we handle the general case as follows: Let $t = t(\epsilon)$ be sufficiently large. Put

$$A_k = \frac{1}{t} \sum_{(k-1)t+1}^{kt} a_u.$$

It is easy to see that $A_k < 2+\epsilon$ for sufficiently large k if $t = t(\epsilon)$ was large enough. To show this, consider

$$S_{kt} - S_{(k-1)t} = 2kt^2 + t^2 + O(kt)$$

$$\geqslant (k-1)\, t\, (a_{(k-1)t+1} + a_{(k-1)t+2} + \ldots + a_{kt}) = (k-1)\, t^2 A_k,$$

or $A_k < 2+\epsilon$ for $t = t(\epsilon)$. Now we obtain from $0 \leqslant a_u < c$ by a simple calculation

$$\sum_{k=1}^{n} A_k\, (A_1 + A_2 + \ldots + A_{n-k} + k)$$

$$= t^{\frac{1}{2}} \sum_{u=1}^{tn} a_u\, (a_1 + a_2 + \ldots + a_{tn-u} + u) + O(n)$$

or

$$\sum_{k=1}^{n} A_k\, (A_1 + A_2 + \ldots + A_{n-k} + k) = n^2 + O(n).$$

Thus since $A_k < 2+\epsilon$ for sufficiently large k, we obtain

$\sum_{k=1}^{n} A_k = n + O(1)$, hence by the definition of the A_k's,

$$\sum_{k=1}^{n} a_k = n + O(1).$$

Thus it will suffice to prove Theorem 2 in case we have

$$0 \leqslant a_k < 2+\epsilon, \text{ for sufficiently large } k. \qquad (15)$$

Define B_n, $\bar{B}_n$ as before. Without loss of generality we can again assume that $B_n = \bar{B}_n$ and that $\bar{B}_n \not\to \infty$, from this last assumption we will obtain a contradiction.

We define the u and v numbers as follows: An integer is a v number if

$$s_v < v - B_n + \log B_n.$$

It is a u number if

$$s_u > u + B_n - (\log B_n)^2.$$

First we have to prove some lemmas about the u and v numbers. Let $\delta = \delta(\epsilon)$ be small but fixed, we have

LEMMA 1. *Let $y < x$. The number $U(x)$ of u numbers not exceeding x satisfies the following inequality:*

$$U(x) - U(y) < \left(\tfrac{1}{2} + \delta\right)(x - y) + o(x).$$

It clearly suffices to prove the lemma if x is a u number, say u_r. We evidently have

$$u_r^2 + O(u_r) = \sum_{k=1}^{u_r} a_k (s_{u_r - k} + k)$$

$$> (u_r + B_n - (\log B_n)^2)(u_r - B_n) + B_n \sum_{k < u_r}{}' a_k, \qquad (16)$$

where the dash indicates that the summation is extended over the k for which $u_r - k$ is a u number (for if $u_r - k$ is a u number $s_{u_r - k} + k > u_r$). From (16) we obtain

$$u_r^2 + O(u_r) \geqslant u_r^2 - B_n^2 - (\log B_n)^2 (u_r - B_n) + B_n \sum_{k < u_r}{}' a_k$$

which implies [$B_n = o(u_r^\epsilon)$ by (9)]

$$\sum{}' a_k = \sum_{j \leqslant r}{}' a_{u_r - u_j} = o(u_r). \qquad (17)$$

But we have from (5)

$$\sum_{k < u_r - y} a_k = u_r - y + o(u_r). \qquad (18)$$

Thus from (17) and (18)

$$\sum_{\substack{k < u_r - y \\ k \neq u_r - u_j}} a_k = u_r - y + o(u_r). \qquad (19)$$

Hence finally from (19) and $a_k < 2 + \epsilon$ we obtain that the sum (19) contains at least $\frac{u_r - y}{2 + \epsilon} + o(v_r)$ summands, or

$$U(u_r)-U(y)=U(x)-U(y)<(\tfrac{1}{2}+\delta)\ (x-y)+o(x),$$

which proves Lemma 1.

LEMMA 2.

$$U(n)>(\tfrac{1}{2}-\delta)n.$$

Denote by m the greatest v number not exceeding n. First we show that $m=n+o(n)$. For, if not, then no integer in the interval $[(1-\epsilon)\ n, n]$ would be a v number. But then

$$S_n=n^2+O(n)=\sum_{k=1}^{n} a_k\ (s_{n-k}+k)\geqslant(n-B_n)\sum_{k=1}^{n} a_k+\log B_n$$

$$\times\sum_{k\leqslant\epsilon n} a_k=n^2-B_n^2+\epsilon n\log B_n+o(n\log B_n),$$

which is clearly false $(B_n=o(n^\epsilon))$. Thus $m=n+o\ (n)$ is proved. Now we have $S_m=m^2+O(m)$

$$=\sum_{k=1}^{m} a_k\ (s_{m-k}+k)\leqslant(m+B_m)\sum_{k=1}^{m} a_k$$

$$-(\log B_n)^2\sum_{k\leqslant m}^{m}{}'' a_k,$$

where in Σ'' the summation is extended over the k's for which $m-k$ is not a u number. Thus

$$m^2+O(m)\leqslant(m+B_n)\ (m-B_n+\log B_n)-(\log B_n)^2\sum_{k<m}{}'' a_k$$

which implies

$$\sum_{m-k\ \text{not}\ u} a_k=o(m). \qquad (18')$$

But since $\sum_{k=1}^{m} a_k=m+o(m)$ and $a_k<2+\epsilon$, the sum $(18')$ can contain at most

$$m-\frac{m}{2+\epsilon}+o(m)$$

summands, which means that $m-k$ is a u number for at least $\frac{m}{2+\epsilon}+o(m)$ values of k, or

$$U(n) = U(m) \geqslant n\ (\tfrac{1}{2} - \delta)$$

which proves the lemma.

Let $z_1, z_2, \ldots,$ be any sequence of integers. The number of z's not exceeding n we denote by $C(n)$. The Schnirelmann density of the z's is defined as the lower bound of $C(n)/n$, If the z's are all less than or equal to m, then we take the lower bound only in the interval $1 \leqslant n \leqslant m$ and we then call it the Schnirelmann density up to m.

LEMMA 3. *Let* $1 < u_1 < u_2 < \ldots < u_x \leqslant n$ *satisfy* $x > c_1 n$. *Then there exists a* u_k *so that the sequence* $u_k + 1 - u_i$, $1 \leqslant u_i \leqslant u_k$ *has Schnirelmann density* $\geqslant c$ *up to* u_k.

REMARK. We assumed $u_1 > 1$, since otherwise the lemma is trivial, we can choose $u_k = u_1 = 1$.

Suppose the lemma is false. Then we can clearly cover the interval from 1 to u_x by intervals of the form $(u_x, u_{y_1}+1), (u_{y_1}, u_{y_2}+1), \ldots, (u_{y_k}, 1)$ so that in each of these intervals the number of u's is less than

$$c_1\ (u_{y_i} - u_{y_{i+1}}),\ i = 0, 1, \ldots k\ (x = y_0,\ 0 = y_{k+1}). \quad (19')$$

Adding the inequalities (19) we obtain

$$C(n) = C(u_x) < c_1\, u_x,$$

an evident contradiction, which proves the lemma.

We define now the u' numbers as follows: An integer $t \leqslant n$ is a u' number if there exists a u number u_r so that $|t - u_r| \leqslant \log B_n$. Let $u'_1 < u'_2 < \ldots < u'_x \leqslant n$ be the sequence of consecutive u' numbers. It is easy to see that the u' numbers satisfy Lemmas 1 and 2. For Lemma 2 this is obvious. It is almost obvious for Lemma 1 too. Because of $0 \leqslant a_k < c$ we evidently have

$$s_{u'} > u' + B_n - (\log B_n)^2 - c \log B_n > u' + B_n - 2\ (\log B_n)^2$$

and Lemma 1 can clearly be proved with $s_{u'} > u' + B_n - 2\ (\log B_n)^2$ instead of $s_u > u + B_n - (\log B_n)^2$.

The main advantage of the u' numbers is that they cannot occur in isolation but always occur in bunches of length not less than $2 \log B_n$ $(B_n \to \infty)$. From

$$s'_{u_1} > u'_1 + B_n - 2(\log B_n)^2$$

we obtain by (5) that $u'_1 > 1$ (in fact $u'_1 = u'_1(n) \to \infty$). Thus Lemma 3 can be applied and we deduce the existence of a u'_y so that the sequence

$$u'_y + 1 - u'_j,\ j \leqslant y$$

has Schnirelmann density $> \frac{1}{2} - \delta$ up to u'_y, also we can assume that u'_y is the largest element of its bunch, thus we immediately obtain that the sequence

$$u'_y - u'_j,\ j \leqslant y$$

has Schnirelmann density $\frac{1}{2} - 2\delta$ up to u'_y.

We have by (17)

$$\Sigma' a_{u'_y - u'_j} = o(u_y). \tag{20}$$

Now we need the following

LEMMA 4. *Let $z_1 < z_2 < \ldots \leqslant z_r \leqslant N$ be a sequence whose Schnirelmann density up to n, c_2 satisfies $\frac{1}{3} < c_2 < \frac{2}{3}$, and for any $m \leqslant N$ the number $C(m)$ of z's not exceeding m satisfy*[5]

$$C(m) < \tfrac{2}{3} m + o(N). \tag{21}$$

Denote further by $y_1, y_2, \ldots y_{N-r}$ the positive integers $\leqslant N$ which are not z's. Then there exists a z, say z_i, so that the sequence $z_j - z_i$ contains more that $N/240$ y's.

Denote by $D(N)$ the number of solutions of

$$z_j - U = y_i,\ 0 < U < N.$$

Clearly to each y_j there are exactly $z_j - j$ U's so that

$$D(N) = \sum_{j=1}^{r} (z_j - j) \geqslant \sum_{j=1}^{n} (4/r.j - j) - r.\,o(N)$$

$$\geqslant \frac{1}{6} r^2 - o(N^2) > \frac{1}{54} N^2 - o(N^2)$$

since from (20) $z_j \geqslant \frac{4}{3} j - o(N)$ and $r > \frac{1}{3} N$. Thus there exists an integer U_0 so that the sequence $z_j - U_0$ contains more than $N/60$ y's.

Next we prove that every integer $\leqslant N$ is the sum of 4 or less z's. By a well-known theorem of Schnirelmann—Landau[6] the Schnirelmann density up to N of the

integers of the form z_i, $z_{j_1}+z_{j_2}$ is $\geqslant 20-c^2 > \frac{2}{3}-\frac{1}{9} > \frac{1}{2}$. Thus clearly every number $\leqslant N$ is the sum of 4 or less z's (since if we have more than $m/2$ a's not exceeding m, then m is the sum of two a's). Write

$$U_o = z_{i_1}+z_{i_2}+z_{i_3}+z_{i_4}$$

(where some of the z's may be o). Denote by E_0 the number of y's in z_j-U_o and by $E_r, r=1,2,3,4$ the number of y's in $z_j-z_{i_r}$, $r=1,2,3,4$.

We now prove that

$$E_o \leqslant E_1+E_2+E_3+E_4. \tag{22}$$

We prove (22) as follows: The sequence $z_j-z_{i_1}-z_{i_2}$ cannot contain more than E_1+E_2 y's for $z_i-z_{i_1}$ contains E_1 y's and from these we obtain not more than E_1 y's of the form $z_j-z_{j_1}-z_{i_2}$. If on the other hand $z_j-z_{i_1}$ is a z, we can get from them at most E_2 y's of the form $z_j-z_{i_1}-z_{i_2}$, (since $z_j-z_{i_2}$ contains E_2 y's). Similarly $z_j-z_{i_1}-z_{i_2}-z_{i_3}$ contains not more than $E_1+E_2+E_3$ z's and $z_j-z_{i1}-z_{i_2}-z_{i_3}-z_{i_4}=z_j-U_0$ contains not more than $E_1+E_2+E_3+E_4$ y's, which proves (22).

Now $E_o > N/60$, hence from (22) we obtain that for some $r \leqslant 4$

$$E_r > N/240$$

which proves Lemma 4.

Consider now the sequence $u'_y-u'_j$. From Lemmas 1, 2 and 3 it follows that Lemma 4 can be applied to it. Thus there exists an u'_k so that the numbers

$$(u'_y-u'_j)-(u'_y-u'_k)=u'_k-u'_j \ (j<k)$$

contain more than $u'_y/240$ positive integers which are not of the form $u'_y-u'_j$. By (17) we get

$$\sum_{j<k} a_{u'_k-u'_j} = o(u'_k) = o(u'_y). \tag{23}$$

Adding (20) and (23) we have

$$\sum_{j<y} a_{u'_y-u'_j}+\sum_{j<k} a_{u'_k-u'_j}=o(u'_y). \tag{24}$$

The sum (24) contains more than

$$(\tfrac{1}{2}-2\delta)\, u'_y+u'_y/240>\left(\tfrac{1}{2}+\frac{1}{300}\right)u'_y \ (\delta \text{ small}) \tag{25}$$

summands. From (5) we have

$$\sum_{r\leqslant u'_y} a_r=u'_y+o(u'_y). \tag{26}$$

From (25) and (26) we obtain

$$\Sigma\, a_r=u'_y+o(u'_y), \tag{27}$$

where in (27) r runs through the integers not occurring in (24). By (25) we see that the sum (27) contains less than $\left(\frac{1}{2}-\frac{1}{300}\right)u'_y$ summands. Thus clearly the equation

$$a_k>2+1/1000$$

would have infinitely many solutions. This is false for $\epsilon<1/1000$. This contradiction establishes Theorem 2, and thus the proof of Theorem 1 is complete.

The question can be raised whether a weaker error term than $O(x)$ in I' suffices to deduce that $\lim \vartheta(x)/x=1$. I can prove that if the error term is $o(x.\log\log x)$, $\vartheta(x)/x\rightarrow 1$ can indeed be deduced. I do not know what the best possible result is in this direction and indeed it is possible that $o(x.\log x)$ suffices for the deduction of $\lim \vartheta(x)/x=1$.

By the methods of the proof of Theorem 2 we can prove the following more general

THEOREM 2′. *Let* $a_k\geqslant 0$, $f(n)>c$, $f(n)/n\rightarrow 0$ *and*

$$\sum_{k=1}^{n} a_k\,(s_{n-k}+k)=n^2+O[n f(n)].$$

Then

$$s_n=n+O(f(n)).$$

It is possible that in Theorem 2 the condition $a_k \geqslant 0$ can be replaced by $a_k > -c$, clearly some condition for a_k is needed.

FOOTNOTES

1. P. Erdös *Proc. Nat. Acad. Sci.* (U.S.A.) (1949), 374-84. See also A Selberg, *Annals of Math.*, (1949) 305-313.

2. Selberg, *ibid.*

3. The fact that *I* implies *II* was known to Selberg before my investigation started (oral communication).

4. If the p_i are primes I'' was proved by Selberg in an elementary way and used for the elementary proof of Dirichlet's theorem *Annals of Math.*, (1949) 297-304.

5. The method used here is very similar to that used in my paper "On the arithmetical density..." *Acta Arithmetica* Vol. 1.

6. E. Landau, *Über einige neuere Fortschritte der Additiven Zahletheorie,* Cambridge Tract No 35, p. 56. HenryMann's proof of the $\alpha+\beta$ hypothesis in *Annals of Math.* (1942) would enable us to deduce that every integer is the sum of 3 z's.

SUPPLEMENTARY NOTE

BY

P. ERDÖS.

[Received 15 November, 1949.]

Theorem 2 of the above paper runs as follows:

Let

$$a_k \geqslant 0, \sum_{k=1}^{n} a_k (s_{n-k}+k) = n^2+O(n) \; (s_m = \sum_{k=1}^{m} a_k). \tag{1}$$

Then

$$s_n = n+O(1). \tag{2}$$

I dealt with this result in a lecture at the University of Illinois this summer and several remarks were made by the audience which I propose to discuss here.

Reiner asked whether anything more can be deduced if in (1) we assume that the error term is $o(n)$. If we put $a_1 = 3/2$, $a_{2k+1} = 2$ for $k > 1$, $a_{2k} = 0$, then $\sum_{k-1}^{n} a_k(s_{n-k}+k) = n^2+o(1)$, but $s_n \neq n+o(1)$. On the other hand if we assume that there exists an $\epsilon > 0$ so that for $k > k_0$, $a_k < 2-\epsilon$, then indeed $\sum_{k=1}^{n} a_k (s_{n-k}+k) = n^2+o(n)$ implies $s_n = n+o(1)$. We do not give the proof since it follows that of the original theorem closely.

Hua raised the following questions: What can be deduced if we assume that $a_k \geqslant 0$ and $\sum_{k=1}^{n} k a_k = \frac{1}{2}n^2+O(n)$, also $a_k \geqslant 0$, and $\sum_{k=1}^{n} a_k (s_{n-k}+k) = \frac{1}{2} n^2+O(n)$?

Here I prove

THEOREM 1. *Let* $a_k \geqslant 0$ *and* $\sum_{k=1}^{n} k.a_k = \frac{1}{2}n^2+O(n)$, *then*

$$s_n = n+O(\log n). \tag{3}$$

and (3) *is best possible.*

To prove (3) put $s_n = n + A_n$. Denote $\max_{m \leqslant n} |A_m| = \bar{A}_n$. We can assume that $\bar{A}_n \to \infty$ (for otherwise (3) holds and there is nothing to prove). Since $\bar{A}_n \to \infty$ we can choose arbitrarily large values of n so that $\bar{A}_n = |A_n|$, and in fact it will be clear from the proof that without loss of generality we can assume $\bar{A}_n = A$. We have

$$\sum_{k=1}^{n} k a_k = n s_n - \sum_{k=1}^{n-1} s_k = n\ (n+\bar{A}_n)$$

$$- \sum_{k=1}^{n-1} (k+A_k) \geqslant \tfrac{1}{2} n^2 + O(n) + \frac{n}{2} (\bar{A}_n - \bar{A}_{n/2}) \quad (4)$$

(if $n/2 < k \leqslant n$ we replace A_k by $\bar{A}_n$, if $k \leqslant n/2$ we replace A_k by $\bar{A}_{n/2}$). If (3) does not hold then clearly $\overline{\lim}\ \bar{A}_{n/\log n} = \infty$, or for every C there exist infinitely many n so that $\bar{A}_n - \bar{A}_{n/2} > C$. But then from (4)

$$\sum_{k=1}^{n} k.a_k > \tfrac{1}{2} n^2 + \frac{C}{2} n + O(n),$$

which contradicts the assumptions of Theorem 1 (since C can be chosen arbitrarily large), which proves (3).

The fact that (3) cannot be improved is immediately clear by putting $a_k = 1 + 1/k$.

THEOREM 2. *Let* $a_k \geqslant 0$, $\sum_{k=1}^{n} a_k s_{n-k} = \tfrac{1}{2} n^2 + O(n)$. *Then*

$$s_n = n + o(n). \quad (5)$$

The error term cannot be $o(n^{1/2})$.

To prove this it suffices to assume that $a_k \geqslant 0$ and $\sum_{k=1}^{n} a_k s_{n-k} = \tfrac{1}{2} n^2 + o(n^2)$. Put $F(x) = \sum_{k=1}^{\infty} a_k x^k$, $F(x)^2 = \sum_{k=1}^{\infty} b_k x^k$. Clearly

$$\sum_{k=1}^{n} b_k = \sum_{k=1}^{n} a_k\, s_{n-k} = \tfrac{1}{2}n^2 + o(n^2).$$

Thus

$$\lim_{x=1} (1-x)^2 F(x)^2 = 1 \text{ or } \lim_{x=1} (1-x)\, F(x) = 1.$$

Hence by the well-known Tauberian theorem of Hardy and Littlewood $s_n = n + o(n)$.

By putting $a_{(n!)^2} = n!$, $a_m = 0$ if $(n!)^2 < m \leqslant (n!)^2 + n!$, $a_m = 1$ otherwise, we immediately obtain that the error term in (5) cannot be $o(n^{1/3})$.

Let $f(x)$ be an increasing function satisfying $f(x) \leqslant x$, $f'(x) \leqslant 1$. $f^{-1}(x)$ is defined by $f[f^{-1}(x)] = x$. Then we have

THEOREM 3. *Let* $a_k \geqslant 0$ *and*

$$S_n = \sum_{k=1}^{n} a_k \left[s_{f^{-1}[f(n)-f(k)]} + f(k)\right] = f(n)^2 + O(f(n)). \quad (6)$$

Then

$$s_n = f(n) + O(1). \quad (7)$$

REMARK: If $f(x) = x$ we obtain our original theorem that (1) implies (2), also $f(x) = x^\alpha$, $0 < \alpha \leqslant 1$, $f(x) = \log x$ satisfy the conditions of Theorem 3.

PROOF OF THEOREM 3. Denote $[f(n)] = N$,

(i.e. $f^{-1}(N) = n + \delta$, $|\delta| < 1$) $\sum_{r < f(n) < r+1} a_k = A_r$. We have from (6)

$$S_{f^{-1}(N+1)} - S_{f^{-1}(N)} = O(N) \geqslant N A_N \text{ or } A_N < c.$$

Thus from (6) by a simple computation, we have

$$\sum_{r=1}^{N} A_r(A_1 + \ldots + A_{N-k_r} + k_r) = N^2 + O(N)$$

which by our theorem clearly implies (7).

ON A CERTAIN RECTILINEAR CONGRUENCE

BY

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[Received 22nd June, 1949.]

1. The pitch of the special congruence formed by lines that meet two given curves Γ ($v =$ const.) and Γ' ($u =$ const.) is given by

$$p = -\iint \frac{\partial^2 x}{\partial u\, \partial v}\, du\, dv,$$

where x is the length of the ray intercepted between the two curves. In this paper a new expression for the pitch is obtained, viz.

$$p = -\iint \frac{\cos \omega - \cos \theta \, \cos \phi \, du\, dv}{x},$$

where ω is the angle between the tangents at two points and θ, ϕ the angles which these tangents make with the ray. It is also proved that such a congruence will be normal only when Γ is a circle and Γ' its axis. Certain other properties of the congruence are also obtained.

2. Let $r(u)$ and $r'(v)$ denote the vector positions of any two points P, P' on the two curves Γ, Γ' and let the surface given by

$$R = \tfrac{1}{2}(r + r')$$

be taken as the director surface so that the equation of the congruence can be written as

$$\mathbf{R} = R + td, \tag{1}$$

where $$d = (r' - r)/x, \text{ or } r' = r + xd. \tag{2}$$

Let also u and v denote respectively the arc lengths of Γ, Γ'. Differentiating (2) with respect to u and v respectively we have

$$0 = t + xd_1 + x_1 d$$
$$t' = xd_2 + x_2 d,$$

where, as usual, the suffixes 1, 2 denote differentiation with respect to u and v respectively. Taking the scalar

product of these two equations with d_1 and d_2, we have in the notation of Weatherburn (*Differential Geometry*, Vol. I)

$$2a+xe=0$$
$$2b'+xf=0$$
$$2b-xf=0$$
$$2c-xg=0.$$

Hence we have $b+b'=0$ and $f(b+b')-ag-ec=0$ and therefore the director surface is the middle surface of the congruence, as is also otherwise evident. The two quadratic forms associated with the congruence are

$$e\,du^2+2f\,du\,dv+g\,dv^2=0$$

and $$e\,du^2 \qquad -g\,dv^2=0.$$

The focal surface degenerates into the two curves $\mathbf{r}, \mathbf{r}'$. Sannia's quadratic form and other forms associated with the congruence can then be easily obtained.

The pitch of the congruence is given by

$$p=-\int(d.\,R_1\,du+d.\,R_2\,dv)$$
$$=-\iint\left\{\frac{\partial}{\partial u}(d.\,R_2)-\frac{\partial}{\partial u}(d.\,R_1)\right\}du\,dv$$
$$=-\iint(d_1.R_2-d_2\,R_1)\,du\,dv$$
$$=-\iint(b-b')\,du\,dv$$
$$=-\iint xf\,du\,dv.$$

Now $d=(r'-r)/x$ and therefore

$$d_1=-\frac{t}{x}-\frac{r'-r}{x^2}\frac{\partial x}{\partial u}$$
$$=-\frac{t}{x}-\frac{d}{x}\frac{\partial x}{\partial u}.$$

Also because $$x^2=(r'-r)^2,$$

$$x\frac{\partial x}{\partial u}=-(r'-r).\,t=-xd.\,t=-x\cos\theta$$

hence $$d_1 = -\frac{t}{x} + \frac{d}{x}\cos\theta.$$

Similarly $$d_2 = -\frac{t'}{x} - \frac{d}{x}\cos\theta.$$

Hence $$e = d_1^2 = \frac{1}{x^2} + \frac{\cos^2\theta}{x^2} - \frac{2\cos^2\theta}{x^2} = \frac{1}{x^2}\sin^2\theta$$

$$g = d_2^2 = \frac{1}{x^2}\sin^2\phi$$

$$f = d_1.d_2 = -\frac{\cos\omega}{x^2} + \frac{\cos\theta\cos\phi}{x^2} + \frac{\cos\theta\cos\phi}{x^2} - \frac{1}{x^2}\cos\theta\cos\phi$$
$$= -(\cos\omega - \cos\theta\cos\phi)/x^2.$$

Therefore

$$p = -\iint \frac{\cos\omega - \cos\phi}{x}\,du\,dv.$$

3. The congruence will be normal only when $b = b' = 0$ and therefore the condition of normality is

$$\cos\omega = \cos\theta\cos\phi,$$

i.e. $$(t.t') - (d.t)\,(d.t') = 0.$$

Writing $s = r' - r = x\,d$, so that $x = |s|$ we get

$$s^2\,(t.t') - (s.t)\,(s.t') = 0,$$

or $$\frac{\partial}{\partial v}\frac{(s.t)^2}{s^2} = 0,\ s.t \neq 0.$$

Hence $$(s.t)^2 = a^2 s^2, \tag{3}$$
where a is a function of v alone. Similarly if $s.t' \neq 0$, $(s.t')^2 = b^2 s^2$, b being a function of v alone. Hence

The curves Γ, Γ′ *are such that each lies on a family of right circular cones with vertices on the other curve and the axes being tangents to the other curve.*

Differentiating (3) w.r.t. u

$$(s.t)\left\{-1 + ks.n\right\} = -a^2 s.t + \frac{a_1}{a}(s.t)^2$$

or $$1 - k\,s.n = a^2 - \frac{a_1}{a}(s.t),\ (s.t \neq 0).$$

Differentiating w.r.t. v

$$-k\, t'.n = -\frac{a_1}{a}(t'.t)$$

i.e. $$t'.\left(k\, n - \frac{a_1}{a} t\right) = 0. \qquad (5)$$

The equation (5) is true for all points on the curves Γ, Γ′ and therefore if u be supposed constant and v to vary, the vector $k\,n - \frac{a_1}{a} t$ is a fixed vector and therefore t' being perpendicular to this vector, the curve Γ′ is a plane curve and therefore being the plane section of a right circular cone is a conic. Now the locus of the vertices of right circular cones that pass through a given conic, consists of two concentric conics in perpendicular planes. Again because the axes of these cones are lines joining the centre to points on the other conic and because it must be a tangent to Γ, Γ must consist of a straight line through the cente of the conic Γ′. It can be easily seen that in that case Γ′ is a circle and Γ its axis.

If $s.t = 0$ we have also $t'.t = 0$ and therefore $k'n'.t = 0$. If $k' = 0$, Γ′ is a straight line and Γ is a plane curve and then Γ is a circle and Γ′ its axis. If $k' \neq 0$ we have $n'.t = 0$ and therefore $\tau' b'.t = 0$ and therefore $\tau' = 0$ and $b' = t' \times n' = t$. Now because b' is a constant vector, Γ is a straight line and then Γ′ is a circle. Hence we have the theorem: *The only normal congrueuce formed by lines that meet two given curves is the one given by transversals to a circle and its axis.*

It follows incidentally that if two curves are such that the distance between any two points on them equals the sum of two functions, one of u and the other of v (the parameters of the two curves), the curves must be a circle and its axis.

SIMPLY ORDERED SPACES*

BY

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[Received 18 June, 1949.]

1. The simply ordered space or the 'chain' with its intrinsic topology has properties analogous with the real number space. Thus it is a normal space and is compact (bi-compact in the notation of Alexandroff and Hopf's '*Topologie*') when it has end elements and every section is a cut. While these properties are mentioned,† proofs of these are not given and we give proofs of these properties making use of a resolution of any open set as a union of maximal 'generalized intervals'. The property of every section being a cut is also shown here to be equivalent to the space being connected.

2. THE OPEN SETS OF THE SIMPLY ORDERED SPACE.

2.1. The simply ordered space consists of a set of points S, linearly ordered by a relation '$<$' (so that for any a, b, c from S either $a<b$, $a=b$ or $a>b$; if $a<b$ and $b<c$, then $a<c$; we write $b>a$ for $a<b$). The topology for the space is defined by assigning as a neighbourhood of any element x of S an *interval* containing x; the interval being defined as a subset of S of one of the types:

(1) all x in $S <$ a fixed element a of S; or

(2) all x in $S >$ a fixed element b of S; or

(3) all x in S such that $b<x<a$ for fixed elements a, b of S.

This neighbourhood system defines S as a Hausdorff space.

* The results of this paper were given before, in a thesis, approved for the M. Sc. Degree of the University of Madras and I am indebted to Dr R. Vaidyanathaswamy for valuable guidance.

† Birkhoff, *Lattice Theory*, p. 28 and Alexandroff and Hopf, *Topologie*, p. 92, Aufgabe.

It may be verified that these intervals are themselves open; they are, of course, the basis of open sets in S.

To get the resolution of open sets into disjoint open sets, we consder the nature of the sections of S. *A section* (A, B) of S is a resolution of S into non-null, disjoint subsets such that every element of A is $<$ each element of B. The section may be one of three types:

(1) A *cut-section*, in which either A has a greatest element (an element $a \geqslant$ every element in A), or B has a least element (an element $b \leqslant$ every element in B), but not both;

(2) a *gap-section*, in which neither A has a greatest element nor B a least element;

(3) a *jump-section*, in which A has a greatest element and B has a least element.

It will be clear that for a cut section (A, B), one and only one of the two section classes A or B is open and the other is closed, while in a jump or gap section both section classes are open and also closed. Thus the space will not be connected if any section were a jump or a gap section. We shall see presently, that conversely the space will be connected if every section were a cut.

In terms of these section classes A, B we define a 'generalized interval' to be

(1) an open lower section-class A of a section (A,B);

(2) an open upper section-class D of a section (C, D);

(3) the intersection $A \cap D$ of an open lower section-class A and an open upper section-class D which are not disjoint.

In terms of these we now prove the resolution theorem:

2.2. THEOREM. 1. *Any open set G of S can be uniquely expressed as the union of disjoint, maximal, generalized intervals* (G_i).

Given an element p of G, we associate two sections (A,B), (C, D) with it as follows:

A consists of all q of $S \leqslant p$ and all q in $G > p$, such that an (open) interval (a, b) contains p, q and is contained in G;

$B = S - A =$ the set of all elements of S not in A.

D consists of all q of $S \geqslant p$ and all q in $G < p$ such that an (open) interval (a, b) contains p, q and is contained in G; $C = S - D =$ the set of elements of S not in D.

It is clear that $(A \cap D)$ is a generalized interval containing p and contained in G. It is also maximal among such; for if $P \cap Q'$ were another, where (P, Q), (P', Q') are sections with P, Q' open and containing p and contained in G, then $r \epsilon P$, r not in A would be possible only if $r > p$ and r not in G. (if $r \epsilon G$ and $r > p$, the union of open intervals containing r, p and contained in G with the open interval (r, p) would give an open interval contained in G and containing r, p so that r would be in A). But then $r \epsilon Q'$ also and so $r \epsilon P \cap Q'$ but not in G, contrary to the hypothesis $P \cap Q' \subset G$. Hence P must be $\subset A$; similarly Q' must be $\subset D$. It is easy to verify that the maximal generalized intervals so associated with distinct points p, q of G are either identical or disjoint. Hence $G = \cup G_i$, where the G_i are the distinct generalized intervals associated with the elements p in G.

NOTES:

(i) When S is the space of real numbers, this resolution will give an utmost ennumerable number of generalized intervals.

(ii) The frontier of a generalized interval $(A \cap D)$ consists of at the most two elements, namely an element $c \leqslant$ every element of the interval and another $b \geqslant$ every element of the interval; the element c exists if (C, D) is a cut with c as the greatest element in C, while b exists if (A, B) is a cut with b as the least element of B.

(iii) The frontier of G contains the frontiers of the components G_i and also the limit points of these frontier points, as $Fr\,G = \bar{G} \cap G'$ is closed. It may contain further points also; for example, let S be the space of rational numbers with the usual order and let G_i be sets of the form ; viz. the rationals lying between $1/\sqrt{n}$ and $1/\sqrt{n+1}$, where n and $n+1$ are two consecutive integers which are not squares, then each $Fr\,G_i = 0$; but $Fr(\cup G_i)$ contains the number 0.

We now pass on to study the topological structure of the space further.

3. Normality and connectedness.

3.1. Theorem 2. *The simply ordered space S is normal.*

That is, given an open set G and a closed set $F \subset G$ we can find an open set H such that $F \subset H \subset \bar{H} \subset G$. To prove this, we first establish a preliminary

Lemma. *If the closed set F is contained in the open set G and $G = \cup G_i$ is the resolution of G into maximal disjoint generalized intervals, then $F_i = F \cap G_i$ is closed and there exists an open set H_i such that $F_i \subset H_i \subset \bar{H}_i \subset G_i$.*

Let $x \in \bar{F}_i - F_i$; then as $\bar{F}_i - F_i \subset \bar{F}_i \subset \bar{F} = F$, $x \in F \subset G$. So there is a generalized interval G_j containing x and contained in G. Then G_j must be identical with G_i. For, otherwise, $x \in \bar{F}_i \subset \bar{G}_i$ and $x \in G_j$ imply $\bar{G}_i \cap G_j \neq 0$, which is impossible for two disjoint generalized intervals G_i and G_j. But if $G_j = G_i$ then also $x \in F$, $x \in G_i$ imply $x \in F_i$, contradicting the hypothesis. Therefore the hypothesis that $\bar{F}_i - F_i$ is not null, is false. Thus $\bar{F}_i \subset F_i$ or F_i is closed. To define H_i we now write $H_i = 0$ if $F_i = 0$; otherwise we set $H_i = P \cap Q$, where P and Q are open sets defined as follows: (G_i is taken to be $A \cap D$, as in Theorem 1).

If G_i contains no elements $>$ every element of F_i, we put $P = A$; if G_i contains some element $p >$ every element of F_i, we set $P =$ set of $q <$ one such p; if G_i contains no

elements < every element of F_i, we put $Q = D$; if G_i contains some $p <$ every element of F_i, we set $Q =$ the set of $q >$ one such p.

Thus in any case P is an open left-section class, Q an open right section class and $H_i = P \cap Q$, a generalized interval. Hence H_i is open and contains F_i and is contained in G_i. If G_i contains no element > every element of F_i, i.e. G_i, F_i are coterminous on the right; hence any limit point of G_i (or P) on the right will also be a limit point of F_i and of P and so belong to F_i and to G_i. Hence the section class P is closed and G_i is closed on the right.

A similar result holds for the left side. Hence in all cases $\bar{P} \subset A$, $\bar{Q} \subset D$ and so $\bar{H}_i = \bar{P} \cap \bar{Q} \subset A \cap D = G_i$. This completes the proof of the lemma.

To prove the main theorem, we now set $H = \cup H_i$, where the H_i's are defined as in the lemma. Then evidently $F \subset H \subset G$. Hence $\bar{H} \subset \bar{G}$. To prove that $\bar{H} \subset G$, we have to show that no point of $\bar{G} - G$ is in $\bar{H}$.

Let $x \in \bar{G} - G = Fr\, G$. As x is not in G, x cannot be in $F (\subset G)$. Then as F is closed, we can find a neighbourhood $U(x)$ of x disjoint with F.

The generalized intervals G_i, none of them contain x and so each one of them is either entirely on the left of x or entirely on the right. We can write $G = \cup G_i + \cup G_j$, where G_i are the intervals to the right of x and G_j to the left of x.

Considering the (G_i) we can find a neighbourhood $w(x)$ of x disjoint with the H_i contained in each of these G_i. This is immediate if a $w(x)$ is disjoint with all the G_i. Otherwise, either every neighbourhood $V(x)$ intersects one G_i, say, G_1, on the right or it is not so for any one G_i. In the former case when $x \in Fr\,(G_i)$, by choosing the $U(x)$ small enough, it can be made disjoint with all the G_i other than G_1, as any G_i is entirely to the right of G_1 or

to the left of G_1 and right of x which would mean every $U(x)$ would not intersect G_1. Then $w(x) = U(x) \cap V(x) \cap \bar{H}_1'$ is a neighbourhood of x disjoint with all H_i on the right. In the second case $V(x)$ will intersect utmost one G_i, say G_1, and contain the G_i (these exist) to the left of G_1, and to the right of x. Taking $W(x) = V(x) \cap \bar{H}_1'$, $w(x)$ is disjoint with the H_i in (G_i) to the right of G_1 while $F_i = F \cap G_i \subset F \cap V(x) = 0$ for the G_i contained in $V(x)$. Hence the corresponding H_i are null and $w(x) \cap H_i$ is null. Thus $w(x)$ is disjoint with the H_i contained in the G_i to the right of x. Similarly we can find a neighbourhood $W'(x)$ of x disjoint with the H_j contained in the G_j to the left of x. Hence $W(x) \cap W'(x)$ is a neighbourhood of x disjoint with $\cup H_i + \cup H_j = H$; hence x is not in $\bar{H}$ when $x \in \bar{G} - G$. Hence $\bar{H} \subset G$. This completes the proof of normality.

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3.2. Next we take up the connectedness of the space.

Theorem 3. *The simply ordered space S is connected if and only if every section is a cut.*

We saw earlier that if some section is not a cut, then the space is not connected. Now let us assume that the space is not connected and shew that there are sections in it that are not cuts. If the space is disconnected, then $S = G + H$, where G and H are both open and are disjoint. Then if $G = \cup G_i$ is the resolution of G into generalized intervals and $G_i = A \cap D$ as in Theorem 1, as G is open and closed also, every G_i is open and closed. Hence $Fr\ G_i$ ($\subset Fr\ G$) is null and $Fr\ A$ is null. Hence as A is an open section class, A is open and closed, so is B then. Therefore (A, B) cannot be a cut section.

3.3. Theorem 4. *A connected simply ordered space S is compact if and only if it contains a least element and a greatest element.*

If S is compact the extreme elements obviously exist. Conversely, assuming the extreme elements a, b to exist (with $a \leqslant$ every element of $S \leqslant b$), we shall find a limit

point for any given infinite subset F of S. For this we define a section (A, B) with B, containing b and all elements q such that only a finite number of elements of F are $> q$. A contains all other elements of S. This clearly defines a section of S. As S is connected, the section must be a cut; so either A or B has an extreme element in the closure of the other. Let x be this point, then evidently $x \in F$. This proves that S is compact if extreme points a, b exist.

We can thus add utmost two points to a connected simply ordered space (if it has no extreme points) without destroying its connectedness.

3.4. THEOREM 5. *The union of two connected simply ordered spaces with a common terminal point is a connected simply ordered space.*

For if, S_1 and S_2 are the connected simply ordered spaces and b their common terminal point, we define an ordering relation in $S_1 \cup S_2$ as follows. Let b be $\geqslant$ every element of S_1 in the order structure of S_1; then choose $<$ as either the ordering relation $(<)$ of S_2 or its reverse $(>)$, according as $b \leqslant$ every element of S_2 or $\geqslant$ every element of S_2. With these definitions of order among pairs of elements in S_1 or S_2 and setting every element of $S_1 <$ every element of S_2. (except when we choose b itself in both, when we set $b = b$), we get the ordering of elements of $(S_1 \cup S_2)$ under which it is a simply ordered space with the topology determined by S_1 or S_2. Also $S_1 \cup S_2$ is connected, since it is the union of two connected spaces with a common point.

ON THE SUM OF A SERIES ANALOGOUS TO FOURIER SERIES.

BY

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1. It is well known that if

$$\bar{a}_n = \frac{1}{\pi}\int_0^{2\pi} f(t) \cos nt\, dt,$$

$$\bar{b}_n = \frac{1}{\pi}\int_0^{2\pi} f(t) \sin nt\, dt \text{ and } f(t \pm 2\pi) = f(t),$$

then the series $\frac{1}{2}\bar{a}_0 + \sum_{n=1}^{\infty} (\bar{a}_n \cos nx + \bar{b}_n \sin nx)$ converges to $f(x)$ under certain conditions. If we substitute for $\bar{a}_n$ and $\bar{b}_n$, the infinite series becomes

$$\frac{1}{2\pi}\int_0^{2\pi} f(t)\, dt + \frac{1}{\pi}\sum_{m=1}^{\infty}\int_0^{2\pi} f(t) \cos m\,(t-x)\, dt$$

which converges to $f(x)$. The question naturally arises: Suppose we have a sequence defined by

$$a_n(x) = \tfrac{1}{2}\int_0^{2\pi} f(t) \cos\left[n\pi \sin \tfrac{1}{2}(t-x)\right] dt,$$

then does the series $\frac{1}{2} a_0(x) + \sum_{n=1}^{\infty} a_n(x)$ converge to $f(x)$? If so, under what conditions?

The object of the present paper is to show that the series converges to $f(x)$ under the same conditions as the Fourier series.

We may note here that

$$a_z(x) = \tfrac{1}{2}\int_0^{2\pi} f(t) \cos\left[z\pi \sin \tfrac{1}{2}(t-x)\right] dt$$

is defined for all real values of z. Since[1]

$$\cos[z\pi \sin \tfrac{1}{2}(t-x)] = J_0(z\pi) + 2\sum_{n=1}^{\infty} J_{2n}(z\pi)\cos n(t-x),$$

we have on multiplying both sides by $f(t)$ and integrating term by term

$$a_z(x) = \tfrac{1}{2}\pi\,[\bar{a}_0(x) J_0(z\pi)] + 2\sum_{n=1}^{\infty} \bar{a}_n(x) J_{2n}(z\pi), \quad (A)$$

provided that term by term integration is permissible, where $\bar{a}_n(x) = \frac{1}{\pi}\int_0^{2\pi} f(t)\cos n(t-x)\,dt$. Again since[2]

$$\int_0^{\infty} J_{2m}(z) J_{2n}(z) z^{-1}\,dz = 0, \ (m \neq n);$$

$$\int_0^{\infty} J_{2n}^2(z)\, z^{-1}\,dz = \tfrac{1}{4}n \ \text{ and}$$

$$\int_0^{\infty} J_{2n}(z)\cos[z\sin\tfrac{1}{2}(t-x)]\, z^{-1}dz = \frac{1}{2n}\cos n(t-x);$$

we have on multiplying (A) by $J_{2n}(z\pi)\, z^{-1}$ and integrating term by term,

$$\int_0^{\infty} a_z(x) J_{2n}(z\pi)\, z^{-1}\,dz = \frac{\pi}{4n}\bar{a}_n(x). \qquad (B)$$

The relations (A) and (B) show that, given the sequence $\{a_n(x)\}$, we can construct the sequence $\{\bar{a}_n(x)\}$ and vice-versa.

2. Let $a_n(x) = \frac{1}{2}\int_0^{2\pi} f(t)\cos[n\pi\sin\frac{1}{2}(t-x)]\,dt$ and $f(x\pm 2\pi) = f(x)$.

Then

$$S_n(x) = \tfrac{1}{2}a_0 + \sum_1^n a_n(x)$$

$$= \tfrac{1}{4}\int_0^{2\pi} \frac{\sin\{(n+\frac{1}{2})\pi\sin\frac{1}{2}(t-x)\}}{\sin\{\frac{1}{2}\pi\sin\frac{1}{2}(t-x)\}} f(t)\,dt.$$

We shall prove that $S_n(x)$ converges to $f(x)$ at any point x under the same conditions as in Fourier Series.

Let $t-x=u$. Then

$$S_n(x) = \tfrac{1}{4}\int_{-x}^{2\pi-x} \frac{\sin\left\{(n+\frac{1}{2})\pi\sin\frac{1}{2}u\right\}}{\sin\left\{\frac{1}{2}\pi\sin\frac{1}{2}u\right\}} f(x+u)\,du$$

$$= \tfrac{1}{4}\left\{\int_{-x}^{0} + \int_{0}^{2\pi} + \int_{2\pi}^{2\pi-x} \frac{\sin\left\{(n+\frac{1}{2})\pi\sin\frac{1}{2}u\right\}}{\sin\left\{\frac{1}{2}\pi\sin\frac{1}{2}u\right\}} f(x+u)\,du\right\}$$

$$= \tfrac{1}{4}\int_{0}^{\pi} \frac{\sin\left\{n+\frac{1}{2})\pi\sin\frac{1}{2}u\right\}}{\sin\left\{\frac{1}{2}\pi\sin\frac{1}{2}u\right\}} \left\{f(x+u)+f(x-u)\right\} du.$$

Let $f(x) = 1$. Then

$$\sigma_n = \tfrac{1}{2}\int_{0}^{\pi} \frac{\sin\left\{(n+\frac{1}{2})\pi\sin\frac{1}{2}u\right\}}{\sin\left\{\frac{1}{2}\pi\sin\frac{1}{2}u\right\}}\,du$$

$$= \tfrac{1}{2}\int_{0}^{\pi}\left\{1+2\cos(\pi\sin\tfrac{1}{2}u)+2\cos(2\pi\sin\tfrac{1}{2}u)+\ldots\right.$$
$$\left.+2\cos(n\pi\sin\tfrac{1}{2}u)\right\}du$$

$$= \tfrac{1}{2}\int_{0}^{\frac{1}{2}\pi}\left\{1+2\cos(\pi\sin v)\right.$$
$$\left.+2\cos(2\pi\sin v)+\ldots+2\cos(n\pi\sin v)\right\}dv$$

$$= \tfrac{1}{2}\pi\left\{1+2J_0(\pi)+2J_0(2\pi)+\ldots+2J_0(n\pi)\right\}.$$

Therefore $\sigma_n \to 1$ as $n \to \infty$, since[3] $\frac{1}{2}\pi+\pi\sum_{1}^{\infty} J_0(m\pi) = 1$.

If we put $\phi(u) = f(x+u)+f(x-u) - 2f(x)$, then we get

$$I_n = S_n(x) - \sigma_n f(x) = \tfrac{1}{4}\int_{0}^{\pi} \frac{\sin\left\{(n+\frac{1}{2})\pi\sin\frac{1}{2}u\right\}}{\sin\left\{\frac{1}{2}\pi\sin\frac{1}{2}u\right\}}\,\phi(u)du.$$

Putting $\frac{1}{2}\pi\sin\frac{1}{2}u = v$, we get

$$I_n = \tfrac{1}{2}\int_{0}^{\frac{1}{2}\pi} \frac{\sin(2n+1)v}{\sin v}\cdot\frac{\psi(v)\,dv}{\left[(\pi/2)^2-v^2\right]^{\frac{1}{2}}} \text{ where } \psi(v) = \phi(u).$$

If $I_n \to 0$ as $n \to \infty$, $\underset{n\to\infty}{\text{Lt}}\, S_n(x) = \underset{n\to\infty}{\text{Lt}}\, \sigma_n f(x) = f(x)$.

Let $$J_n = \frac{1}{\pi}\int_0^{\frac{1}{2}\pi} \frac{\sin (2n+1) v}{\sin v} \psi(v)\, dv;$$

then

$$I_n - J_n = \tfrac{1}{2}\int_0^{\frac{1}{2}\pi} \frac{\sin (2n+1) v}{\sin v} \psi(v) \left\{\frac{1}{[(\pi/2)^2 - v^2]^{\frac{1}{2}}} - \frac{2}{\pi}\right\} dv.$$

Since

$$\frac{\psi(v)}{\sin v}\left\{\frac{1}{[(\pi/2)^2 - v^2]^{\frac{1}{2}}} - \frac{2}{\pi}\right\} \text{ is summable in } \left(0, \frac{\pi}{2}\right)$$

it follows by the usual Riemann-Lebesgue theorem that $I_n - J_n \to 0$ as n tends to infinity on the *sole assumption* that $f(x)$ is summable in $(0, \frac{1}{2}\pi)$ and is periodic. Hence $S_n(x) \to f(x)$ if $J_n \to 0$, i.e. if

$$\underset{n\to\infty}{\text{Lt}} \int_0^{\frac{1}{2}\pi} \frac{\sin (2n+1) v}{\sin v} \psi(v)\, dv \to 0.$$

Now J_n differs from the partial sums $\bar{S}_n$ of the Fourier series of $f(x)$ in that $\psi(v)$ replaces

$$f(x+2v) + f(x-2v) - 2f(x) = \phi(2v).$$

From the relation connecting u and v $(v = \frac{\pi}{2}\sin\frac{1}{2}u)$, all the usual conditions for $\bar{S}_n$ to tend to zero will make $J_n \to 0$ and vice-versa.

3. Summability of the series.

Let us write $$\sigma_n = \frac{S_0 + S_1 + \ldots + S_{n-1}}{n}$$

$$= \frac{1}{4n}\int_0^{\pi} \frac{\left\{\sin(\frac{1}{2}\pi \sin\frac{1}{2}u) + \sin(\frac{3}{2}\pi\sin\frac{1}{2}u) + \ldots + \sin(\frac{1}{2}\,\overline{2n-1}\,\pi\sin\frac{1}{2}u)\right\}\left\{f(x+u) + f(x-u)\right\}}{\sin(\frac{1}{2}\pi\sin\frac{1}{2}u)} du$$

$$= \frac{1}{4n}\int_0^{\pi} \frac{\sin^2(\frac{1}{2}n\pi\sin\frac{1}{2}u)}{\sin^2(\frac{1}{2}\pi\sin\frac{1}{2}u)}\left[f(x+u) + f(x-u)\right] du.$$

In the particular case when $f(x) = 1$, we get

$$\bar{\sigma}_n = \frac{1}{2n}\int_0^{\pi} \frac{\sin^2(\frac{1}{2}n\pi\sin\frac{1}{2}u)}{\sin^2(\frac{1}{2}\pi\sin\frac{1}{2}u)} du.$$

Now

$$\bar{S}_n(x) = \tfrac{1}{2}\int_0^\pi \frac{\sin\{(n+\frac{1}{2})\pi\sin\frac{1}{2}u\}\,du}{\sin\{\frac{1}{2}\pi\sin\frac{1}{2}u\}}, \text{ when } f(x)=1$$

$$= \tfrac{1}{2}\pi + \pi\,[J_0(\pi)+\ldots+J_0(n\pi)].$$

Hence

$$\bar{S}_0+\bar{S}_1+\ldots+\bar{S}_{n-1} = \tfrac{1}{2}\pi n + \pi\,\{(n-1)J_0(\pi)+(n-2)J_0(2\pi)$$
$$+\ldots+(n-\overline{n-1})J_0(\overline{n-1}\,\pi)\}.$$

Therefore, when $f(x)=1$,

$$\bar{\sigma}_n = \frac{\bar{S}_0+\bar{S}_1+\ldots+\bar{S}_{n-1}}{n}$$

$$= \tfrac{1}{2}\pi + \pi\,\{J_0(\pi)+J_0(2\pi)+\ldots+J_0(\overline{n-1}\,\pi)\}$$
$$-\frac{1}{n}\pi\{J_0(\pi)+2J_0(2\pi)+\ldots+(n-2)J_0(\overline{n-1}\,\pi)\}$$

As $n\to\infty$, the second limit tends to zero and hence $\bar{\sigma}_n \to 1$ Therefore

$$\bar{I}_n = \sigma_n - \bar{\sigma}_n S = \frac{1}{4n}\int_0^\pi \frac{\sin^2(\frac{1}{2}n\pi\sin\frac{1}{2}u)}{\sin^2(\frac{1}{2}\pi\sin\frac{1}{2}u)}\phi(u)\,du.$$

Let $\frac{1}{2}\pi\sin\frac{1}{2}u = v$. Then we define

$$\bar{J}_n = \frac{1}{2n}\int_0^{\frac{1}{2}\pi}\frac{\sin^2(nv)}{\sin^2 v}\psi(v)\,dv.$$

Therefore

$$\bar{I}_n - \bar{J}_n = \frac{1}{2n}\int_0^{\frac{1}{2}\pi}\frac{\sin^2 nv}{\sin^2 v}\psi(v)\left(\frac{1}{[(\pi/2)^2-v^2]^{\frac{1}{2}}}-\frac{2}{\pi}\right)dv.$$

Now

$$|\bar{I}_n - \bar{J}_n| \leqslant \frac{1}{2n}\int_0^{\frac{1}{2}\pi}\frac{1}{\sin^2 v}\psi(v)\left(\frac{1}{[(\pi/2)^2-v^2]^{\frac{1}{2}}}-\frac{2}{\pi}\right)dv$$

and the integral on the right exists. Therefore $\bar{I}_n - \bar{J}_n \to 0$ as $n\to\infty$.

Therefore the condition for $(C, 1)$ summability becomes $\underset{n\to\infty}{\mathrm{Lt}}\ \frac{1}{n}\int_0^{\frac{1}{2}\pi}\frac{\sin^2 nv}{\sin^2 v}\psi(v)\,dv = 0$, where as before $\psi(v)$

replaces $\phi(2v)$. Hence all the theorems on $(C, 1)$ summability of Fourier series follow immediately.

My best thanks are due to the referee for his very valuable suggestions.

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